One, Two or Many Frequencies: Synchrosqueezing, EMD and Multicomponent Signal Analysis

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with thanks to
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Some caveats….

- Mathematics in presentation minimised….
- If you want to explore the methods you will <u>need</u> read the Mathematics described in the references!
- Don't use the code without reading the papers!

Structure

- Introduction
- Signals, Time and Frequency
- Empirical Mode Decomposition
- Reassignment
- SynchroSqueezing
- Summary

What is a signal?

A **signal** is a time (or space) varying quantity that can carry information.

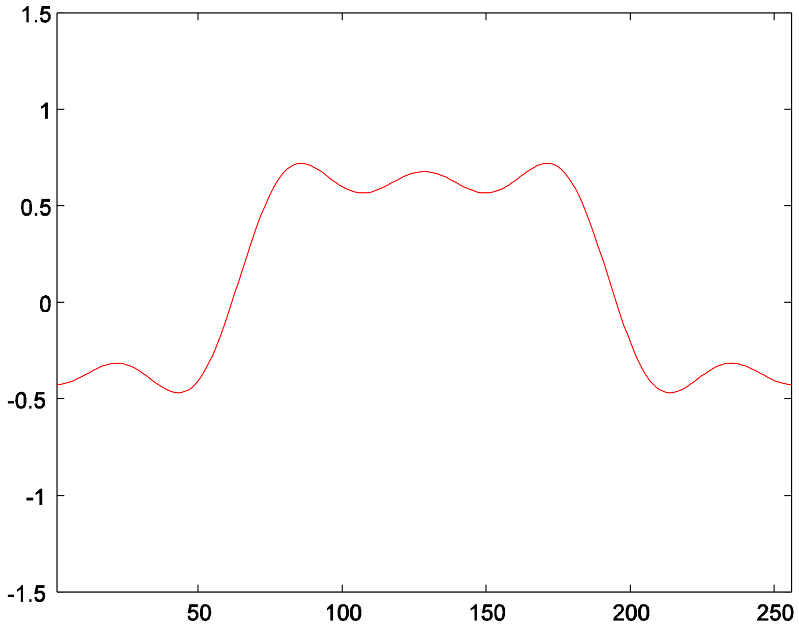
The concept is broad, and hard to define precisely.

(*Wikipedia*)

How do we represent signals?

The Frequency viewpoint (Fourier):

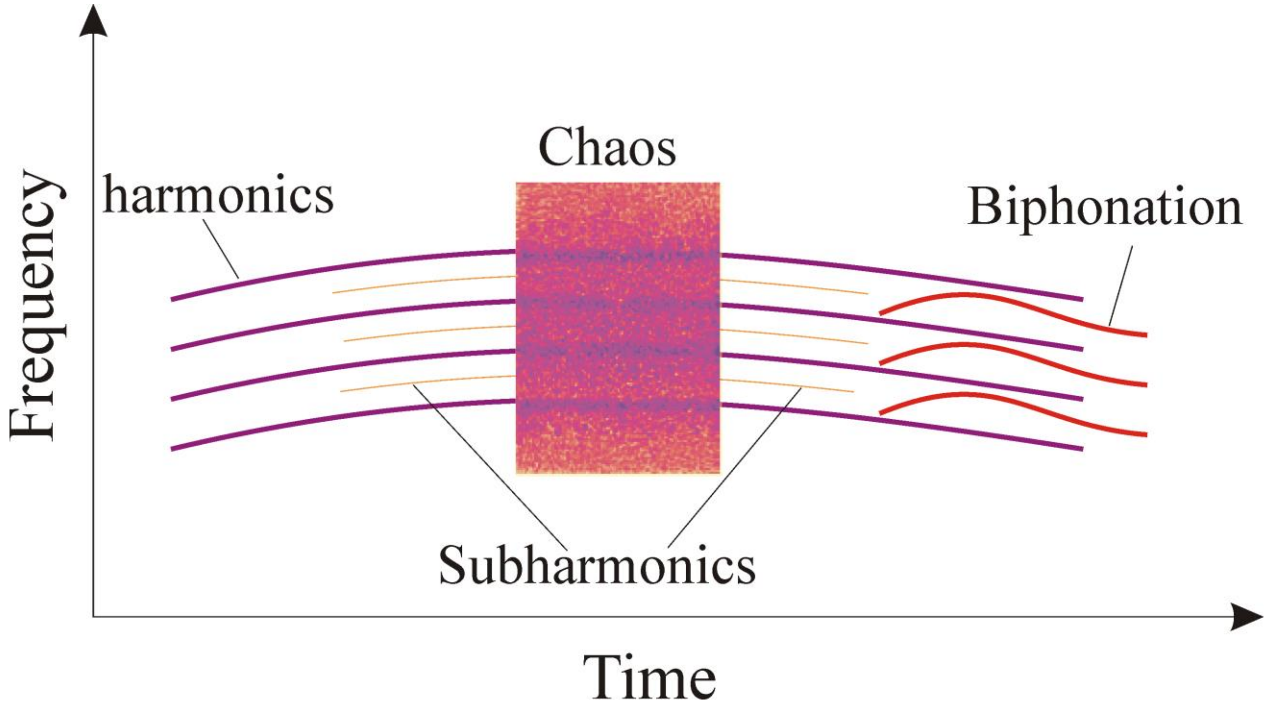
Signals can be built from the sum of harmonic functions (sine waves)



Joseph Fourier

$$\underbrace{x(t)}_{\text{signal}} = \sum_k \underbrace{c_k}_{\text{Fourier coefficients}} \underbrace{\phi_k(t)}_{\text{Harmonic functions}}$$

What is a Signal?



W. Tecumseh Fitch, “Calls out of chaos: the adaptive significance of nonlinear phenomena in mammalian vocal production”, Animal Behaviour, 2002

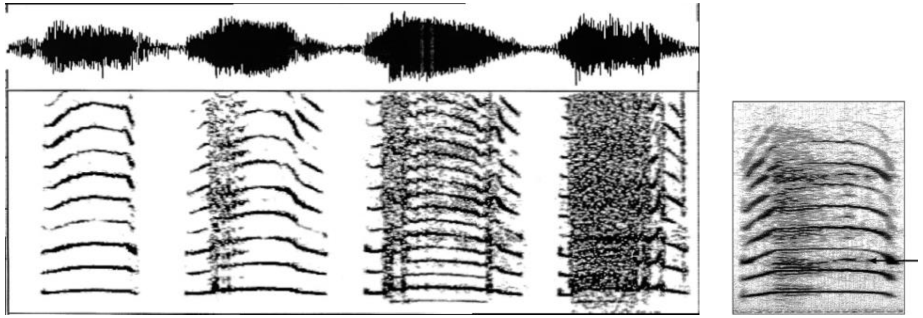
What is frequency?

Frequency is the number of occurrences of a repeating event per unit time. It is also referred to as temporal frequency. The period is the duration of one cycle in a repeating event, so the period is the reciprocal of the frequency.

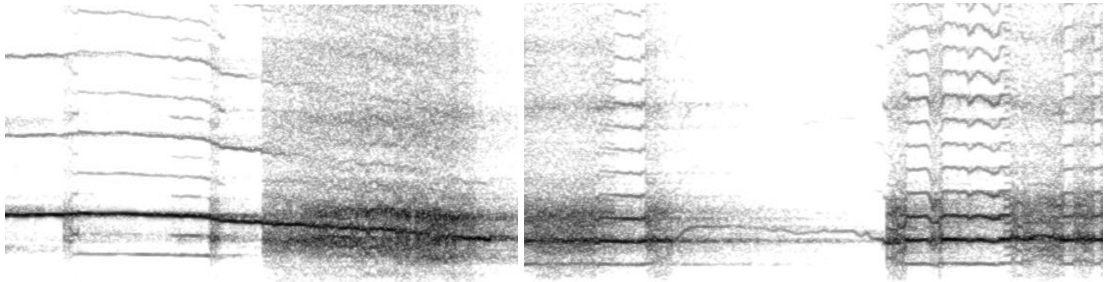
(Wikipedia)

What is frequency?

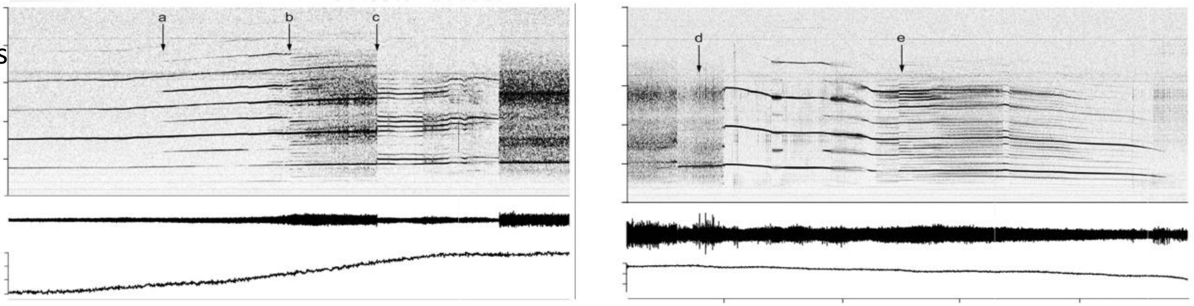
Macaque



Fatima Miranda

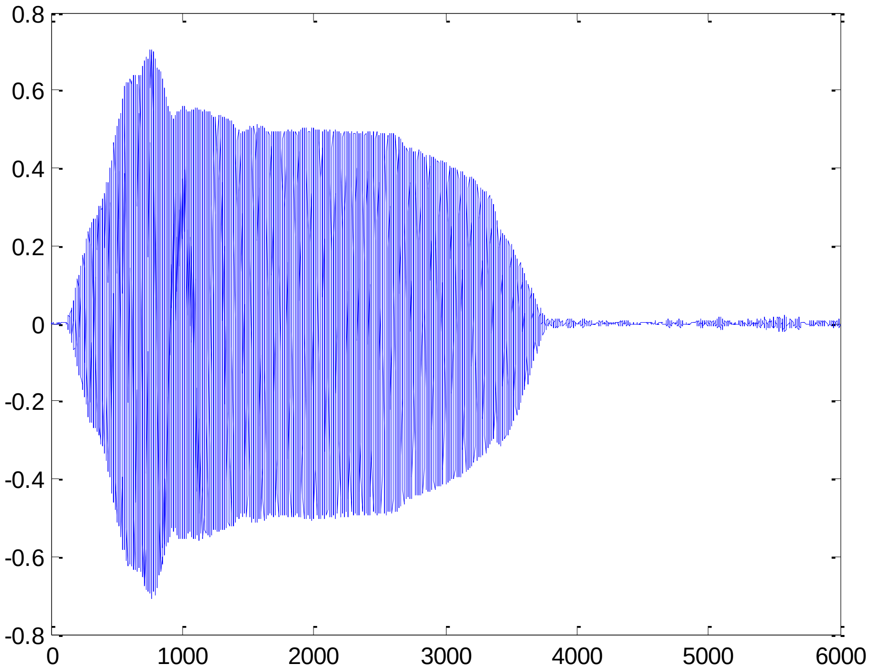


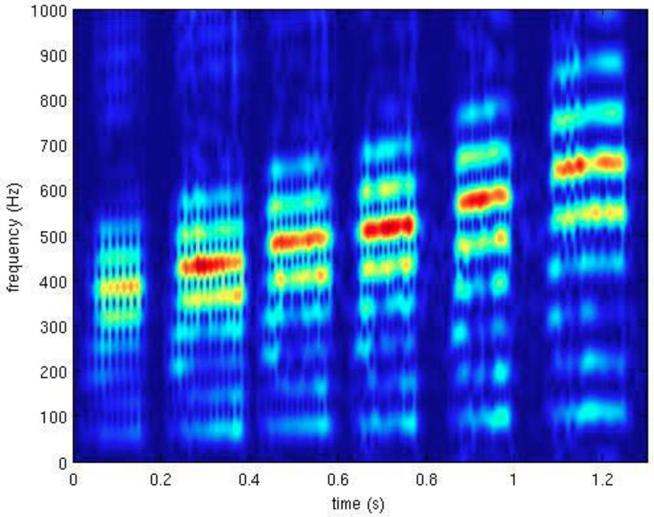
Amolops tormotus



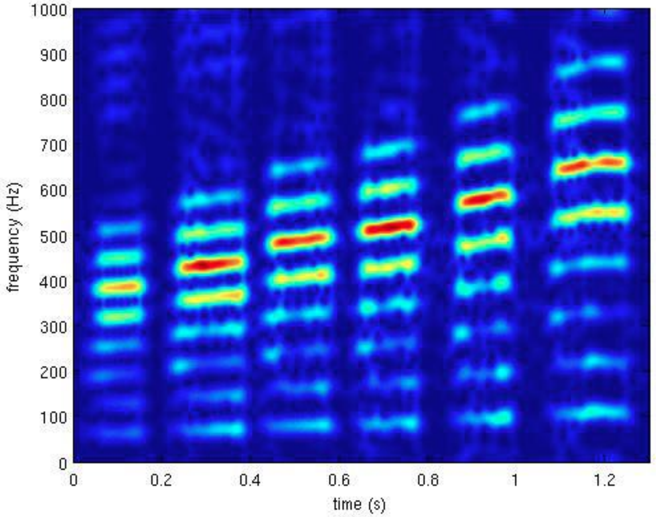
What is a Signal

Frequency modulation (FM) implies the instantaneous frequency varies. This contrasts with amplitude modulation, in which it is the amplitude which is varied while its frequency remains constant. Many signals can be modeled as a sum of AM and FM components.

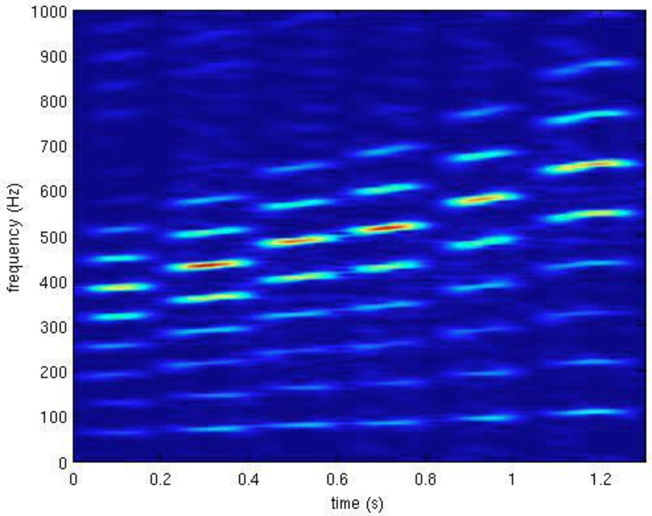




frequency (Hz)
time (s)



frequency (Hz)
time (s)



frequency (Hz)
time (s)

Signal Analysis

- Fourier Based Methods
  - Relatively Robust
  - No assumptions about the data
  - Outside the data window the data is assumed to be zero
    - Results in distortion
    - Windowing also leads to distortions
- Modern Spectral Analysis
  - Parametric
    - Assumes an underlying signal model (AR, MA, ARMA)
    - Eliminates the need for windows
  - Non-Parametric
    - Eigen based analysis
    - Problems with non-white noise
    - Poor in dealing with broadband processes

Empirical Mode Decomposition

Basic idea is very simple:

signal = fast oscillation + slow oscillation

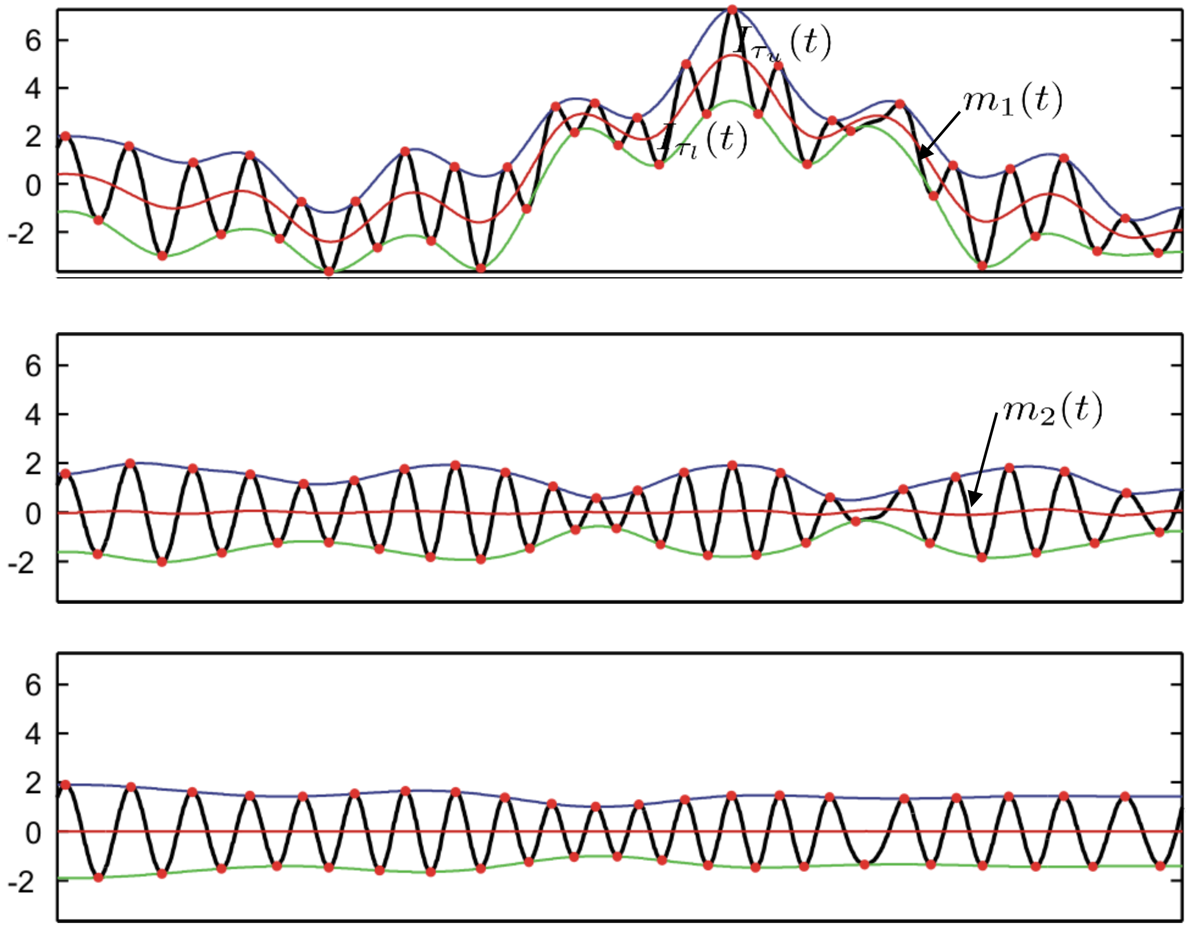
(& iterations)

- Separation “fast vs slow’ data driven
- Local analysis based on neighbouring extrema
- Oscillation rather than frequency

Empirical Mode Decomposition

- Identify local maxima and local minima
- Deduce an upper envelope and a lower envelope by interpolation (cubic splines)
    - subtract the mean envelope from the signal
    - iterate until “mean envelope = 0" (sifting)
    - subtract the obtained mode from the signal
    - iterate on the residual

Empirical Mode Decomposition: Sifting



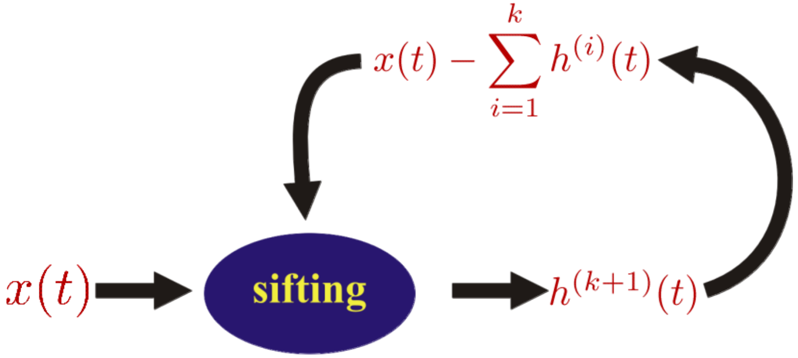
$$m_1(t) = \frac{I_{\tau_u}(t)+I_{\tau_l}(t)}{2}$$

$$\hat{h}_1^{(1)} = x(t) - m_1(t)$$

$$\hat{h}_{500}^{(1)} = \hat{h}_{499}^{(1)}(t) - m_{500}(t)$$

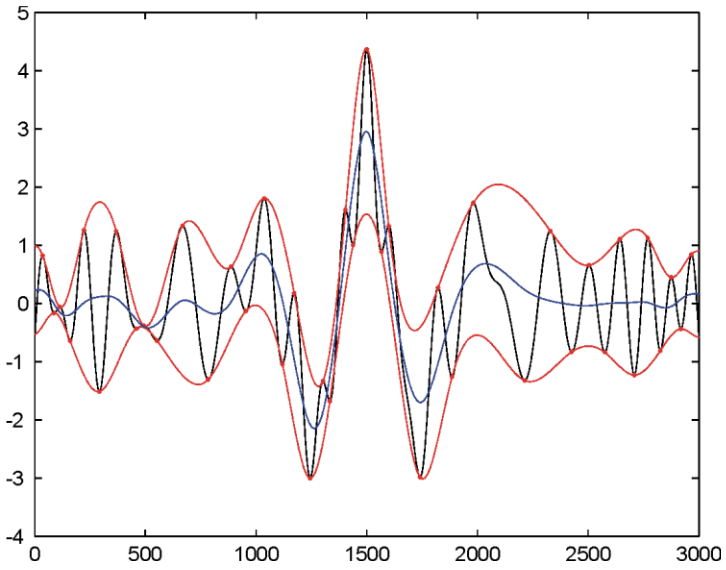
Empirical Mode Decomposition

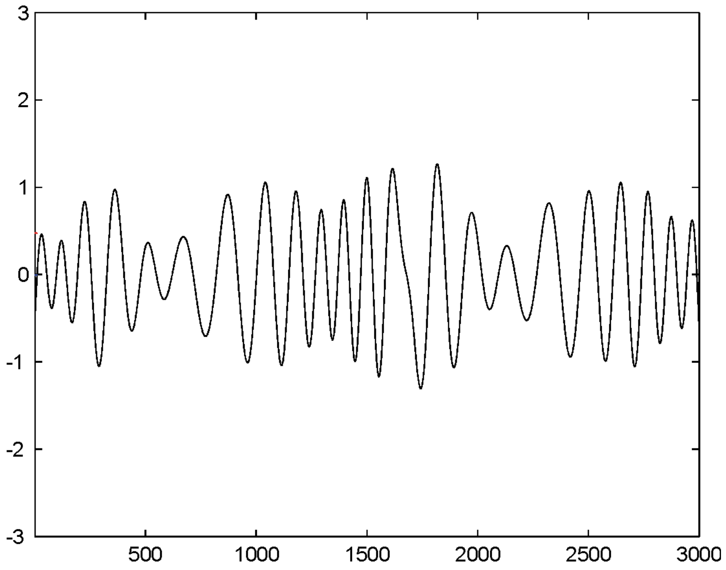
- **Decomposition:** $x(t) = \sum_{i=1}^{N} h^{(i)}(t) + r(t)$
- **Mode:** Intrinsic Mode Functions (IMF's) - Represents the oscillation modes embedded in the data. Usually, $h(t) = a(t)\cos(\omega(t))$



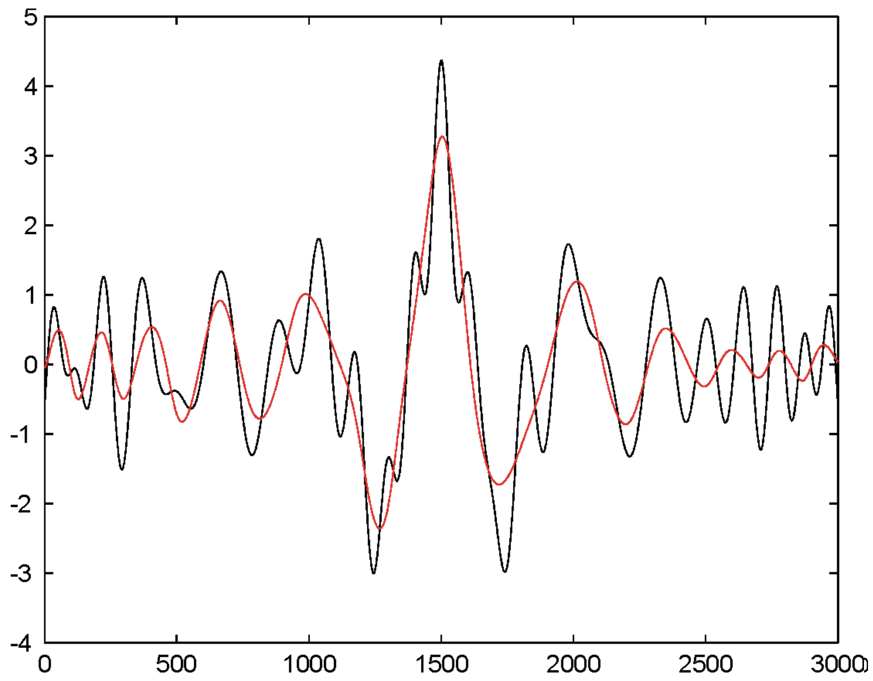
- **Empirical:** The Sifting process is essentially defined by an algorithm. EMD **lacks any theoretical foundations**.

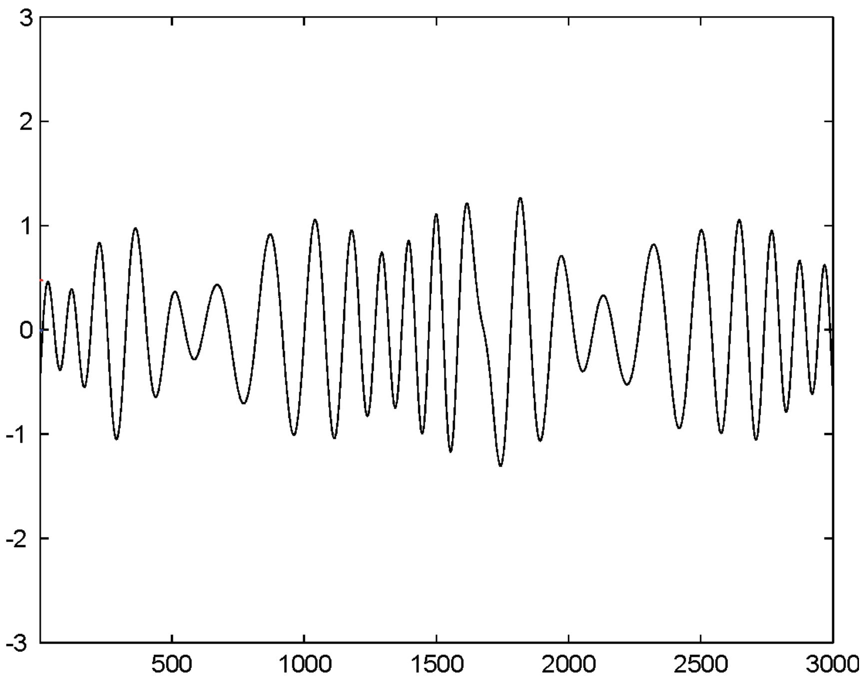
Sifting Process





Sifting Process





Efficient way of computing the local mean

- Zero mean
- Consists of a succession of maxima and minima where all maxima are positive and all minima are negative.
- The frequency locally is higher than the local mean signal

Beating

- When a signal is composed of two components with close instantaneous frequencies, EMD exhibits a beating phenomenon [Rilling and Flandrin (2008)].
- More precisely, if a signal is composed of two harmonics, i.e., $f(t) = \cos(2\pi t)+a\cos(2\pi\xi_0 t)$, *[* Rilling and Flandrin 2008] show an interesting zone in the *a vs. ξ plane (amplitude vs.* frequency plane) where EMD misidentifies the sum of two components as only a single component; the precise shape of this zone depends on the value ξ_0.
- *They also* quantified this phenomenon and termed it *beating, because EMD "identified"* the two harmonics as a single oscillating (i.e., beating) signal.

EMD - Summary

- Locality — The method operates at the scale of one oscillation
- Adaptive — The decomposition is fully data-driven
- Multi-resolution — The iterative process explores sequentially the “natural” scales of a signal
- Oscillations of any type — No assumption on the nature of oscillations (e.g., harmonic) ⇒ 1 nonlinear oscillation = 1 mode
- No analytic definition — The decomposition is only defined as the output of an algorithm ⇒ analysis and evaluation ?

Hilbert Spectrum

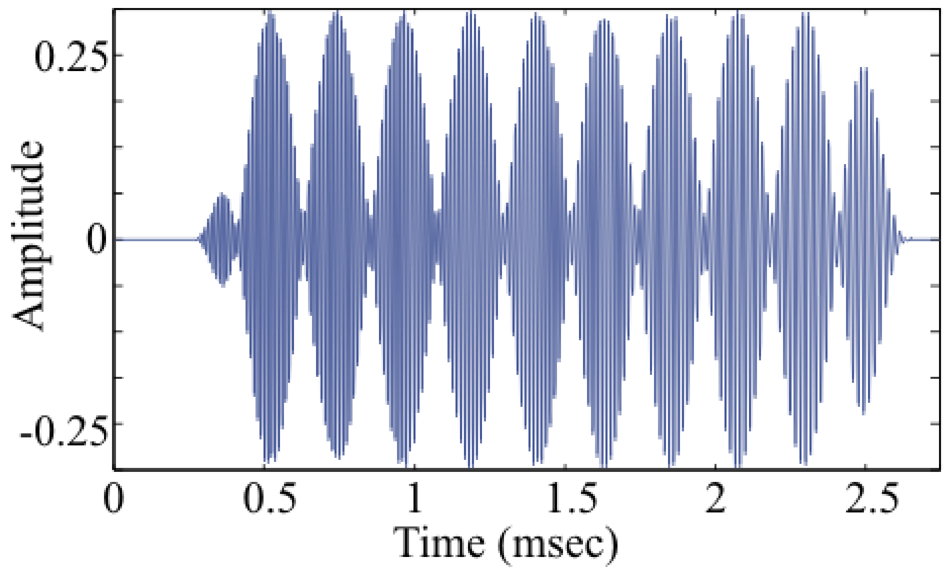
- Can we produce TF-representation plots based on the IF, IA estimates?

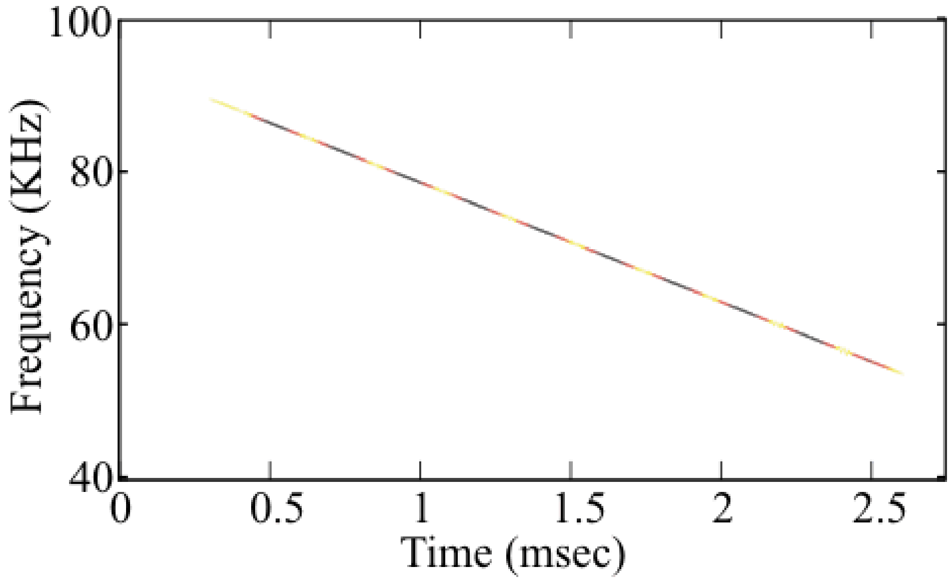
- Hilbert Spectrum: IA and IF are combined in order to form a spectrogram-like plot, $H(t,f)$.

- Specifically, for any time instant t, $H(t,f)=\alpha(t)$ if IF(t) equals to f and zero at the rest of the frequencies.

- In practice, a Time – Frequency grid is adopted in order to construct $H(t,f)$.

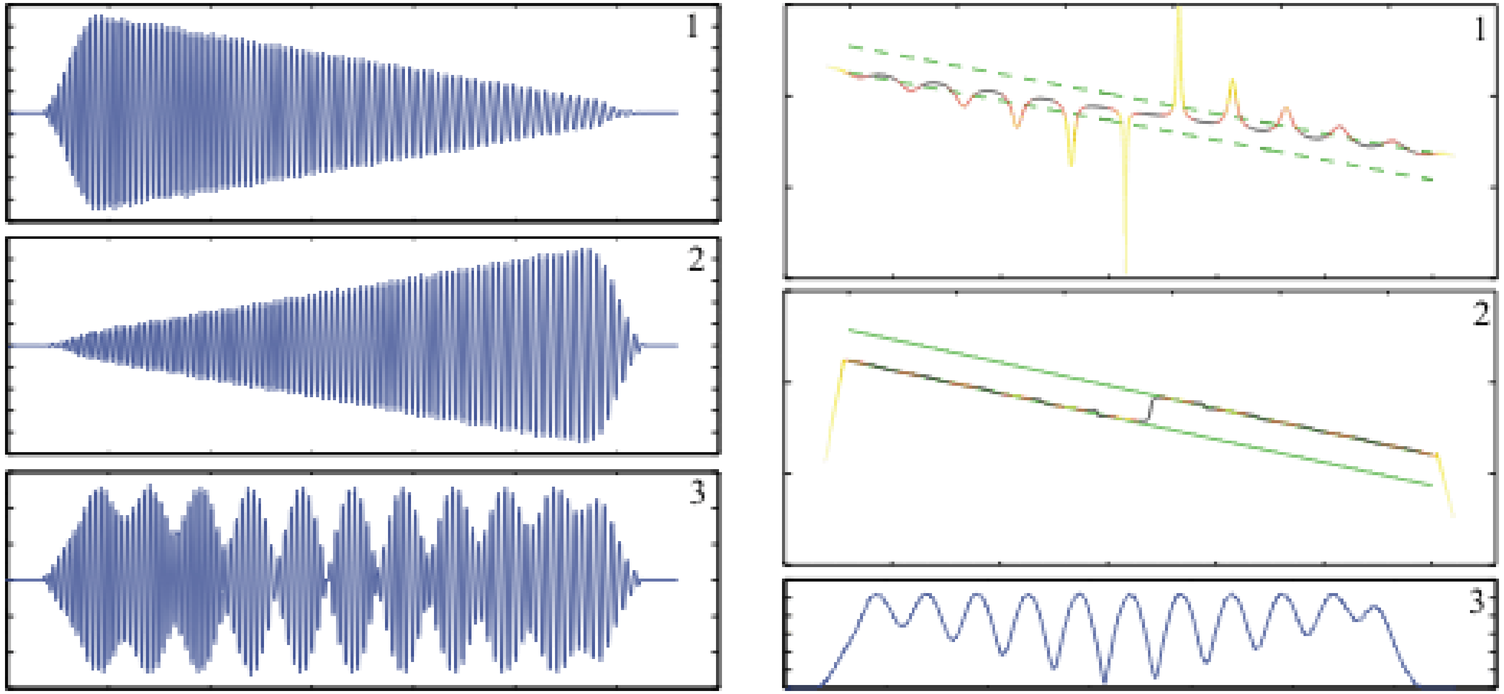
On Instantaneous Frequency: Mono-component signals





In the case of mono-component signals, IF coincides with the actual frequency of the signal.

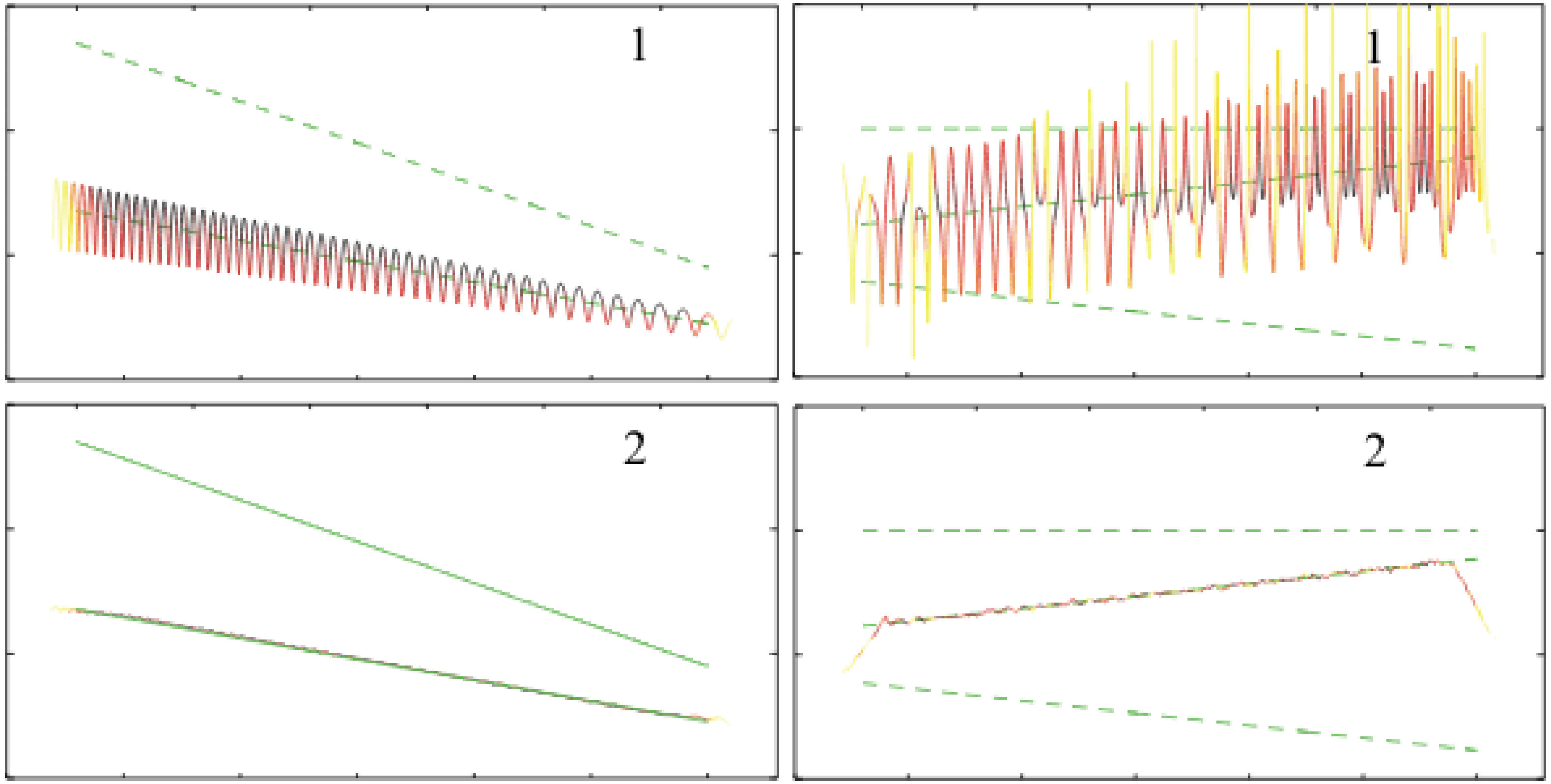
Instantaneous Frequency Multi-component signals



(a) (b)

- IF is a non-symmetric oscillation exhibiting spikes that point toward the component with the larger amplitude.
- The strength of the IF oscillations (that is the height of the spikes) depends on the relative amplitudes of the components.

Instantaneous Frequency Multi-component signals



(a) (b)

The further apart in frequency the components are, the larger the amplitude of oscillations become.

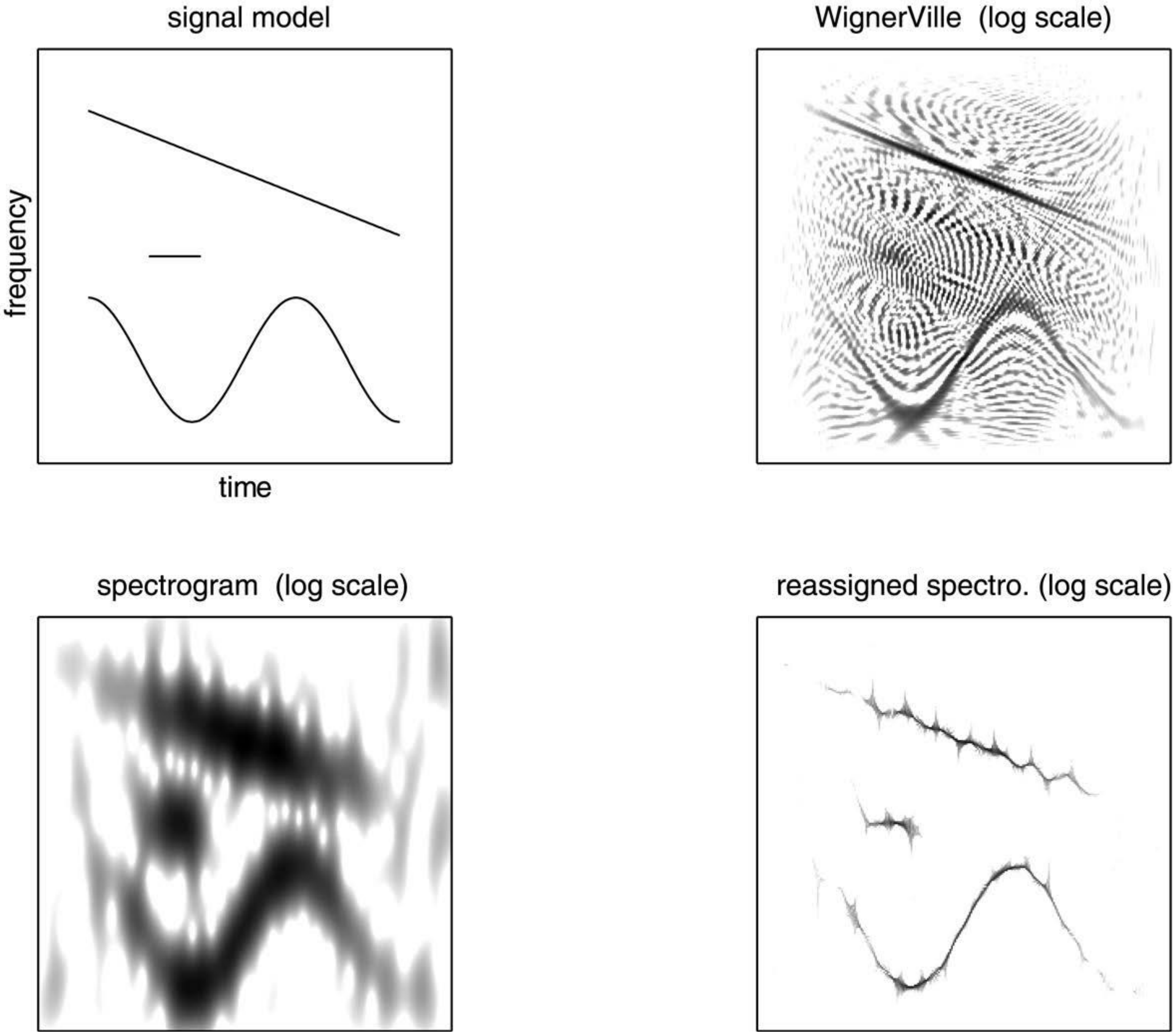
The frequency of oscillations is relative to the frequency difference.

Time Frequency Reassignment

- Time-Frequency Reassignment
  - Originates from a study of the STFT, which smears the energy of the superimposed Instantaneous Frequencies around their center frequencies in the spectrogram.
  - TFR analysis “reassigns” these energies to sharpen the spectrogram

Wigner Ville Methods

A variety of methods were developed using Wigner-type distributions, tailored to guarantee localization of signals with specific underlying FM laws, though at the expense of cross-terms that are problematic in the multi-component case.



From Auger and Flandrin 2002

Reassignment basics

A Spectrogram $S_x^h(t,\omega)$ usually defined as $S_x^h(t,\omega) := \left|F_x^h(t,\omega)\right|^2$ where

$$S_x^h(t,\omega)=\int_{-\infty}^{\infty} x(s)h^*(s-t)e^{-i\omega s}ds \times e^{i\omega t/2}$$

Note the dependence on the window function *h(t)* which aims to limit evaluating on a specific neighbourhood of time *t.* In essence the window is a measurement device which could of course depend adaptively on the signal.

Reassignment basics

A natural choice for *h(t)* is the matched filter, i.e. take a time reversed version of the signal. Thus

$$F_x^{x^-}(t,\omega) = \frac{W_x\left(\frac{t}{2},\frac{\omega}{2}\right)}{2}$$

Where

$$W_x(t,\omega) := \int_{-\infty}^{\infty} x\left(t+\frac{s}{2}\right) x^*(t-\frac{s}{2})e^{-is\omega}\, ds$$

Which is the Wigner Ville Distribution (WVD).

Can be perfectly localized for linear FM signals;

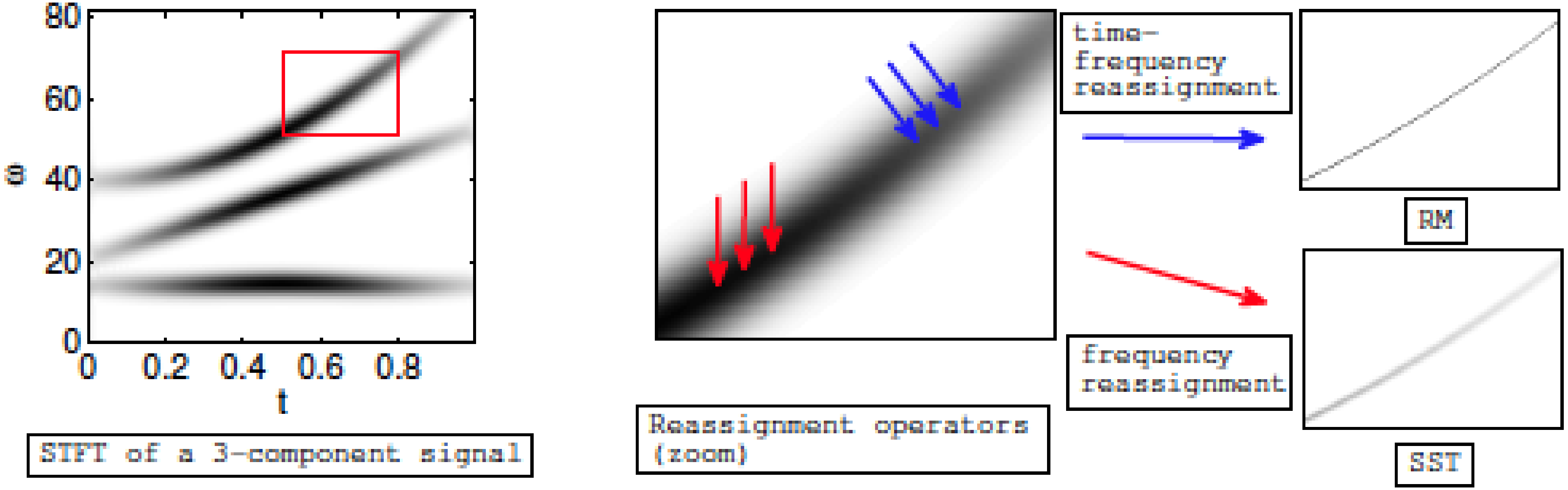
Cannot be positive everywhere, thus forbidding a local density interpretation.

Quadratic nature creates spurious cross terms, characterized by oscillating contributions located midway in between any two interacting components

Reassignment basics

- So a Spectrogram is essentially a smoothed WVD!
- That is the value of the spectrogram at a given point *(t,ω)* of the plane results from the summation of a continuum of WVD contributions within some TF domain defined as the essential TF support of the short-time window.
- In essence, the distribution of values is summarized by a single number, and this number is assigned to the geometric center of the domain over which the distribution is considered.

Reassignment example



center: blue (resp. red) arrows symbolizing how reassignment is performed with RM (resp. SST) on a small patch (delimited by the red segments) extracted from the 3- component signal STFT depicted on the left; right: reassignment carried out with RM (resp. SST) for the signal STFT depicted in the central subfigure

SynchroSqueezing

- Synchrosqueezing
  - Special case of reassignment methods [Auger and Flandrin (1995); Chassande–Mottin *et al. (1997, 2003*)].
    - SSQ reallocates the coefficients resulting from a continuous wavelet transform based on the frequency information, to obtain a concentrated picture over the time-frequency plane, from which the instantaneous frequencies are then extracted
    - Adaptive to the signal;
    - Reconstruction of the signal from the reallocated coefficients is feasible
    - Solid Theoretical foundations

SynchroSqueezing

- Idea from Daubechies & Maes 1996
  - Concentrate (squeeze) wavelet coefficients, at fixed times, on the basis of local frequency estimation (synchronised)
    - Guarantees a sharply localised representation
    - Allows for reconstruction of identified modes
    - Offers a mathematical tractable alternative to EMD

Synchrosqueezing – Reconstruction Approach

We are interested in the retrieval of the components f_k of a multi-component signal f defined by:

$$f(t) = \sum_{k=1}^{K} A_k(t) \cos\left(2\pi\phi_k(t)\right) = \sum_{k=1}^{K} f_k(t)$$

The problem can be viewed from two perspectives

1. Compute an approximation of the modes, either by using EMD or via wavelet projections.
2. Use the SST to reallocate the WT before proceeding with multicomponent retrieval

Retrieval of Multiple Components (RCM):

Practical Implementation (Brevdo et al 2011)

The strategy developed by Brevdo et al for the RCM is to proceed on a component by component basis. The idea is to find a curve, $(C^*_q)_{q=0,\ldots,n-1}$,in the time-frequency plane such that it maximizes the energy which forces the modes to be smooth through a total variation penalization term and is obtained by computing the following quantity:

$$c^* = \underset{c\in\{0,\cdots,n_a-1\}^n}{\arg\max} \sum_{q=0}^{n-1} \log\left(\left|T_{d,f}(\omega_{c_q}, q)\right|^2\right) - \sum_{q=1}^{n-1} \lambda\Delta\omega\left|c_q - c_{q-1}\right|^2$$

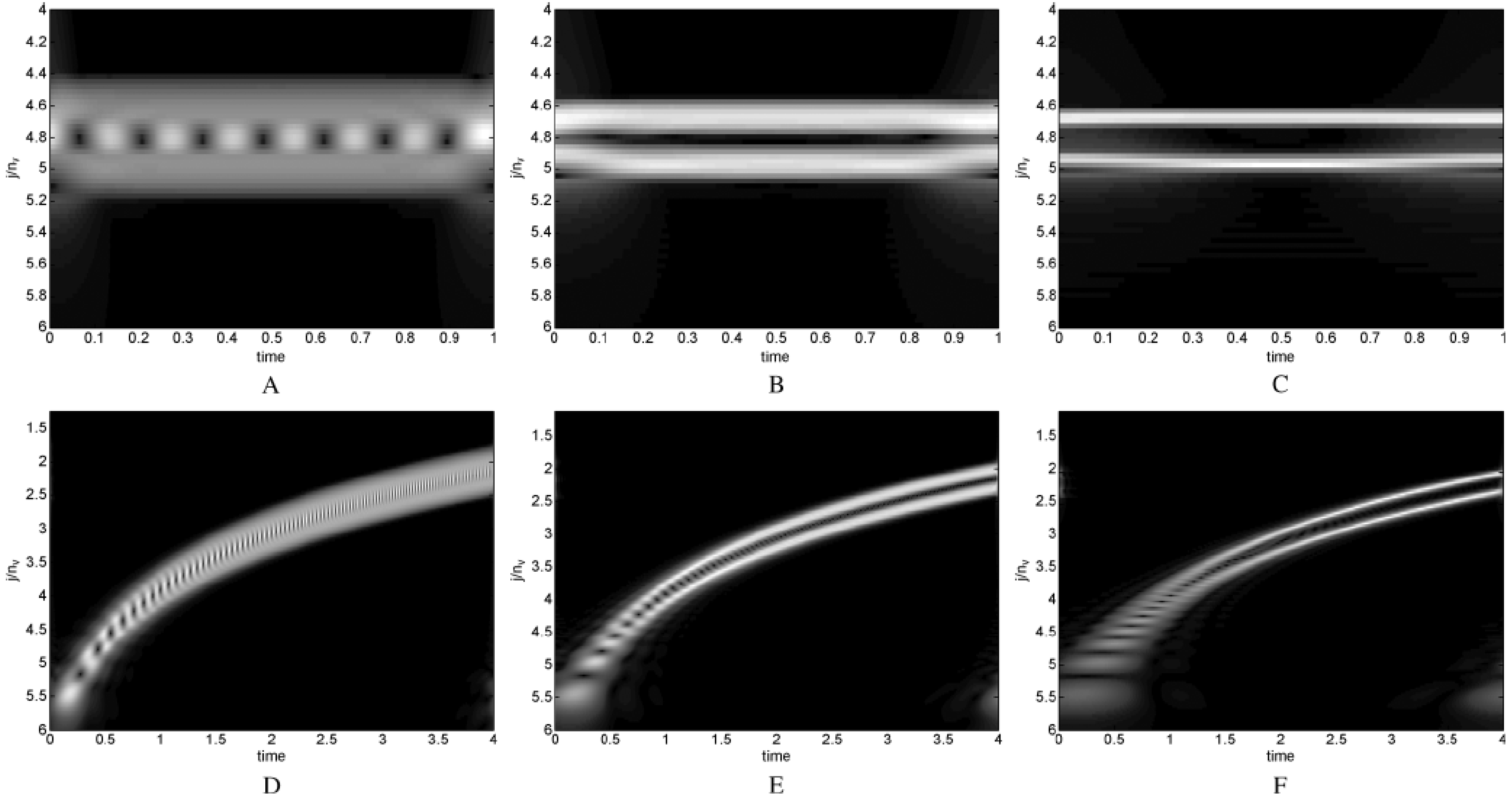


Fig. 1. (A): Modulus of the WT of $\cos(80\pi t) + \cos(65.44\pi t)$ computed using the bump wavelet with $\mu = 1, \sigma = 0.2$; (B): idem as (A) taking $\mu = 1$ and $\sigma = 0.1$; (C): idem as (A) taking $\mu = 1$ and $\sigma = 0.05$; (D): modulus of the WT of $\cos(60\pi(t + {}^1/_4)^2) + \cos(49.10\pi(t + {}^1/_4)^2)$ using the bump wavelet taking $\mu = 1$ and $\sigma = 0.2$; (E): idem as (D) taking $\mu = 1$ and $\sigma = 0.1$; (F): idem as (D) taking $\mu = 1$ and $\sigma = 0.05$.

Summary so far

- Auger & Flandrin coined the term reassignment and showed that the explicit use of the STFT phase can be efficiently replaced by a combination of STFTs with suitable windows.
- Maes & Daubechies then developed synchrosqueezing which is a special case of reassignment, with the additional advantage of enabling reconstruction.
- The STFT of an AM–FM component or mode spreads the information relative to that mode in the TF plane around a curve commonly called a ridge. Conventionally, the focus of signal reconstruction has been on dealing directly with these ridges.
- Consider, an alternative view by considering a mode as a particular TF domain which we term a basin of attraction

Reassignment

- Time-Frequency Reassignment (TFR) provides a sharpened spectrogram display based on locally calculated instantaneous time and frequency in the STFT
- Spectrogram energy is reassigned (moved) from
  - *(t,f)* = current-frame-time, current-bin-frequency) to
  - *(t',f')* = (instantaneous-time, instantaneous-frequency)
- where *(t',f')* are computed locally from the spectrogram in a neighborhood about *(t,f)*

Reassignment Basics

The aim of RM is to compensate for the TF shifts induced by the two-dimensional smoothing which defines the STFT. To do so, a meaningful TF location to which the local energy given by the spectrogram is assigned. This corresponds to the centroid of the distribution, whose coordinates are efficiently computed

$$\hat{t}_f(\omega, t) = t + \mathrm{Re}\left\{ \frac{V_f^{tg}(\omega, t)}{V_f^{g}(\omega, t)} \right\}$$

$$\hat{\omega}_f(\omega, t) = \omega - \mathrm{Im}\left\{ \frac{V_f^{g'}(\omega, t)}{V_f^{g}(\omega, t)} \right\},$$

Reassignment Vector (RV)

$$RV(\omega, t) = \nabla \log |V_{fg}(\omega, t)|$$

The expression above suggests a relationship between RV and local extrema of the amplitude of the STFT: $RV(\omega,t)$ is the null vector if and only if $|V_{fg}(\omega, t)|$ (ie the magnitude of the STFT) admits a local extremum along both time and frequency directions.

This indicates that RVs tend to point towards local maxima that can be interpreted as attractors, whereas zeros act as repellers.

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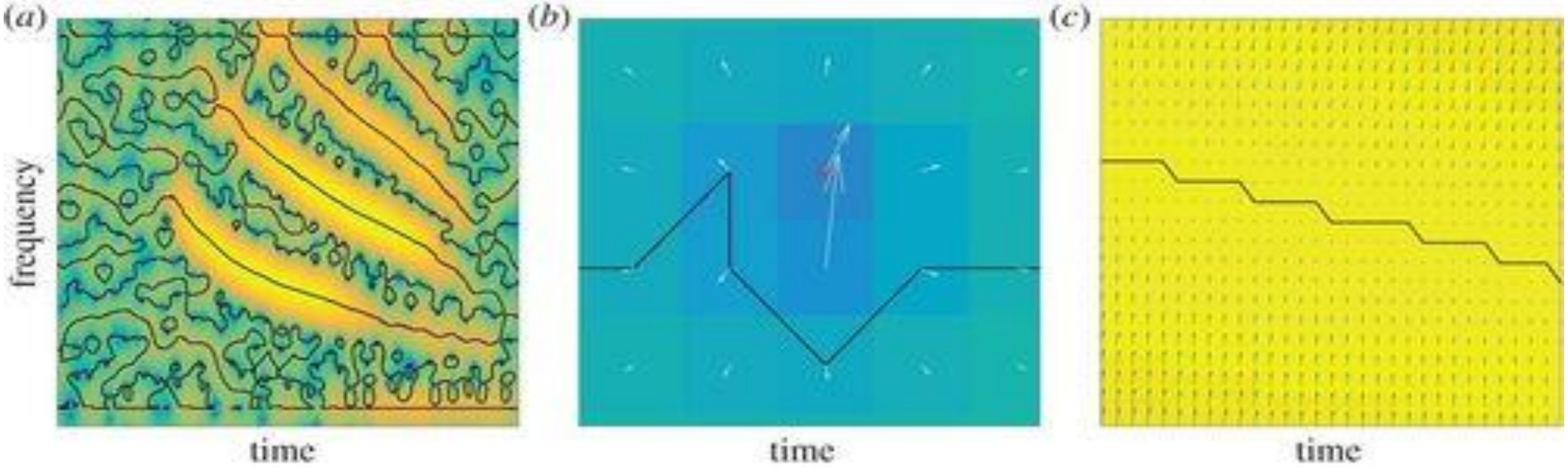
On the Zeros of the STFT

- Rather than considering time and frequency independently, it can be interesting to consider them as the real and imaginary parts of a complex-valued variable, thus identifying the TF plane with the complex plane.
- After some manipulation this can be shown to correspond to the Bargmann factorisation of the STFT and with the help of the Weierstrass–Hadamard theorem it can be shown that the STFT is fully characterised by its zeros.

Two Approaches to Mode Reconstruction

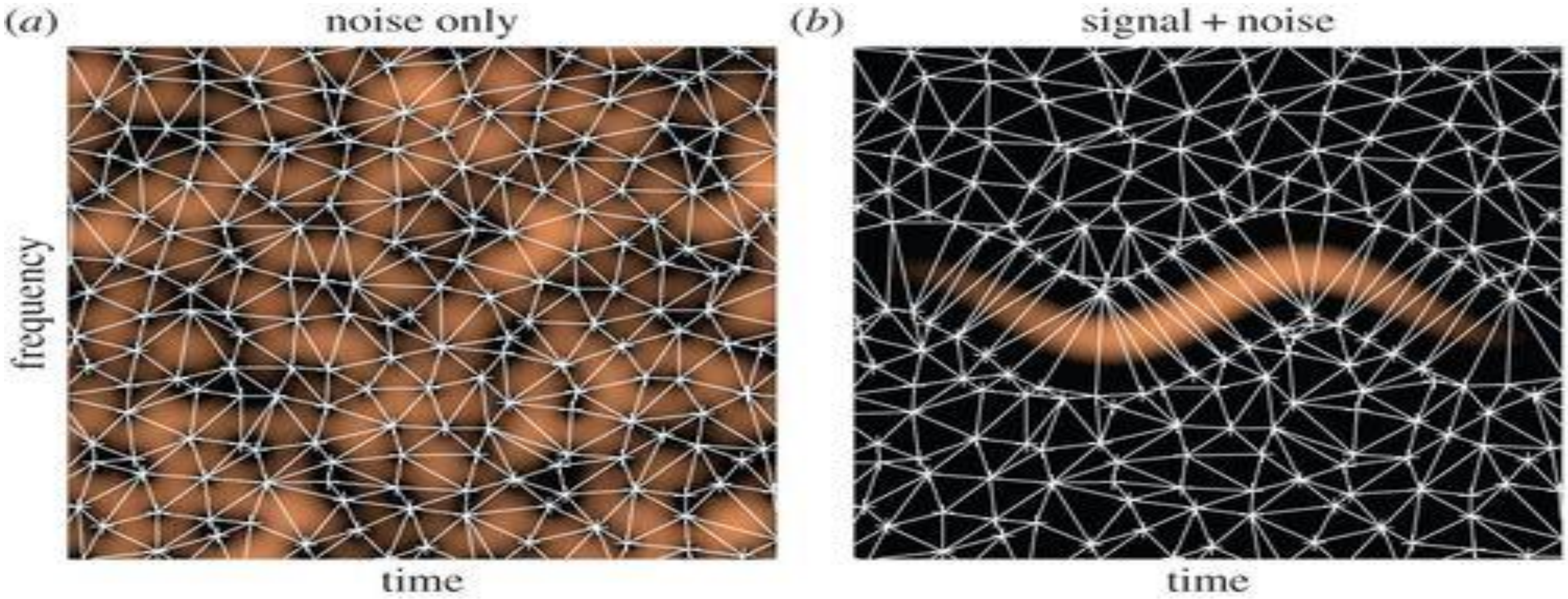
- Determine the ridge associated with a mode by considering either
  - Local maxima of the spectrogram in some predefined direction or
  - Zeros of the ridge points, in relation to reassignment techniques.
- The technique used to determine the basin of attraction is derived directly from the method for ridge extraction.
- The STFT of a signal is fully characterized by its zeros and then exploits the distribution of these zeros for Gaussian noise to deduce an algorithm that computes the mode domains.
- Mode reconstruction is then carried out by simply integrating the information inside these domains.

Approach 1: Determination of basins of attraction based on ridges and reassignment vectors



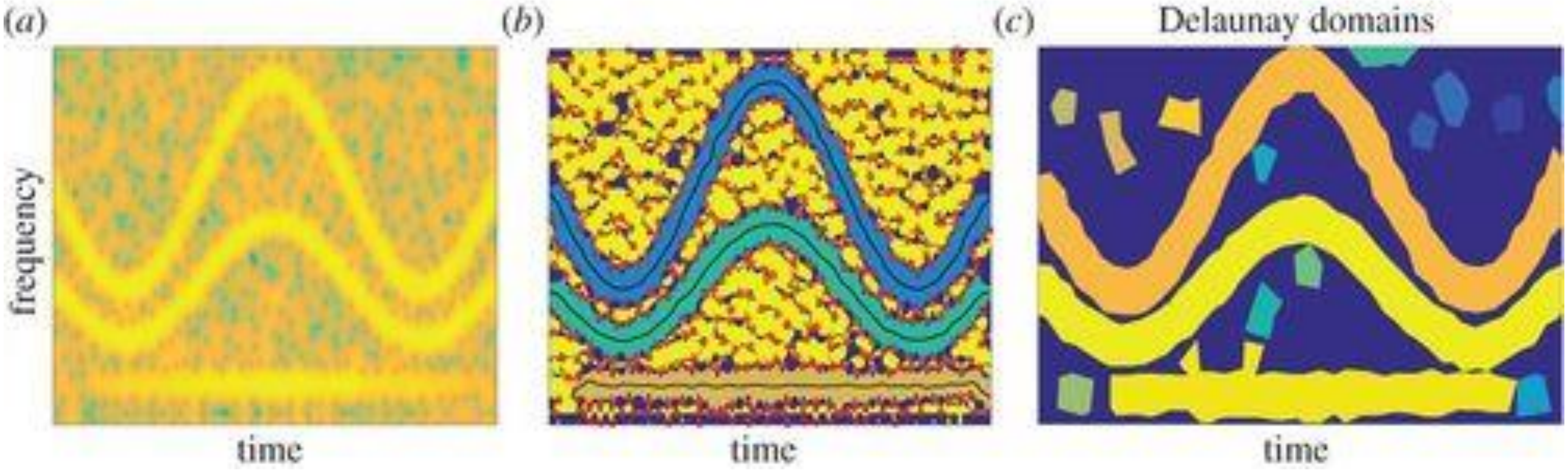
The figure represents (a) Signal spectrogram of a bat signal with the zeros and the ridges superimposed. In (b) the RV close to a zero of the spectrogram (red asterisk); white arrows represent the RV and the nearby contour is plotted in black.(c) RV close to a ridge of the spectrogram; blue arrows represent the RV and the ridge is plotted in black. (Online version in colour.)

Approach 2: Determination of mode domains based on Delaunay triangulation upon zeros



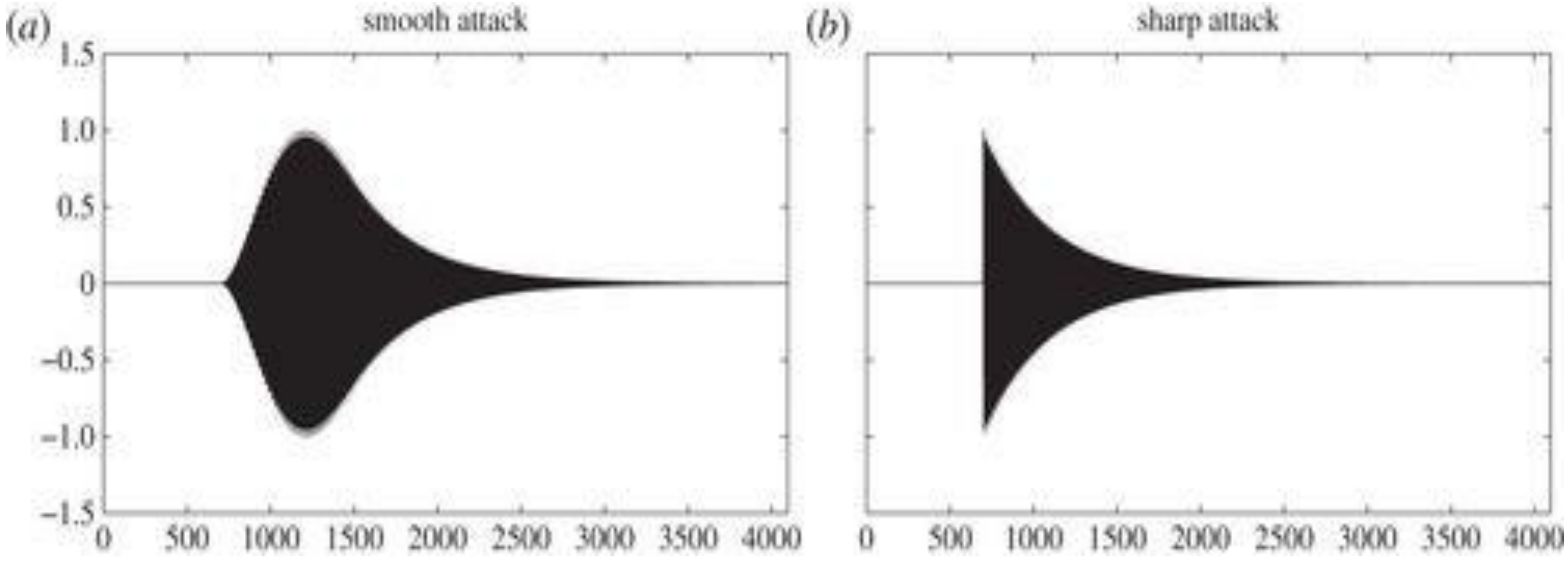
The figure shows (a) Delaunay triangulation based on the zeros of the spectrogram for a noise signal. (b) Delaunay triangulation based on the zeros of the spectrogram of a mode superimposed onto the noise of the left diagram.

Mode Reconstruction

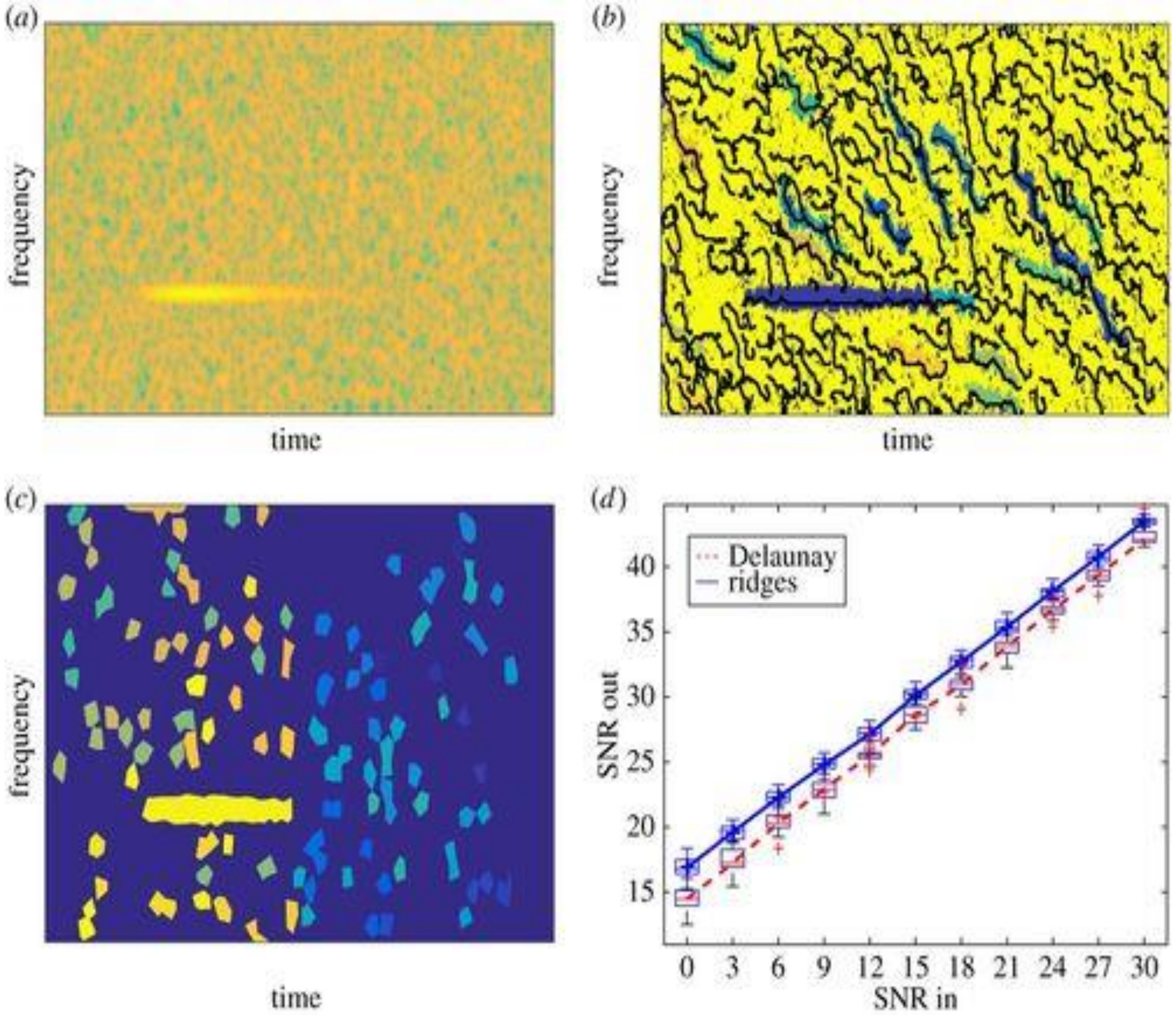


(a) Spectrogram of synthetic three-mode signal with additive Gaussian noise (SNR=10 dB). (b) Ridges and basins of attraction (c) Mode domains computed using Delaunay triangulation.

Numerical Validation



Damped tone with (*a*) a smooth attack (raised cosine) and (*b*) a sharp attack (step function).



In this figure: (a) Spectrogram of a noisy damped tone with a smooth attack (SNR= 0 dB). (b) Main ridges and basins of attraction derived from the RV. (c) Mode domains derived from Delaunay triangulation. (d) Denoising performance of the mode reconstruction. (Online version in colour.)

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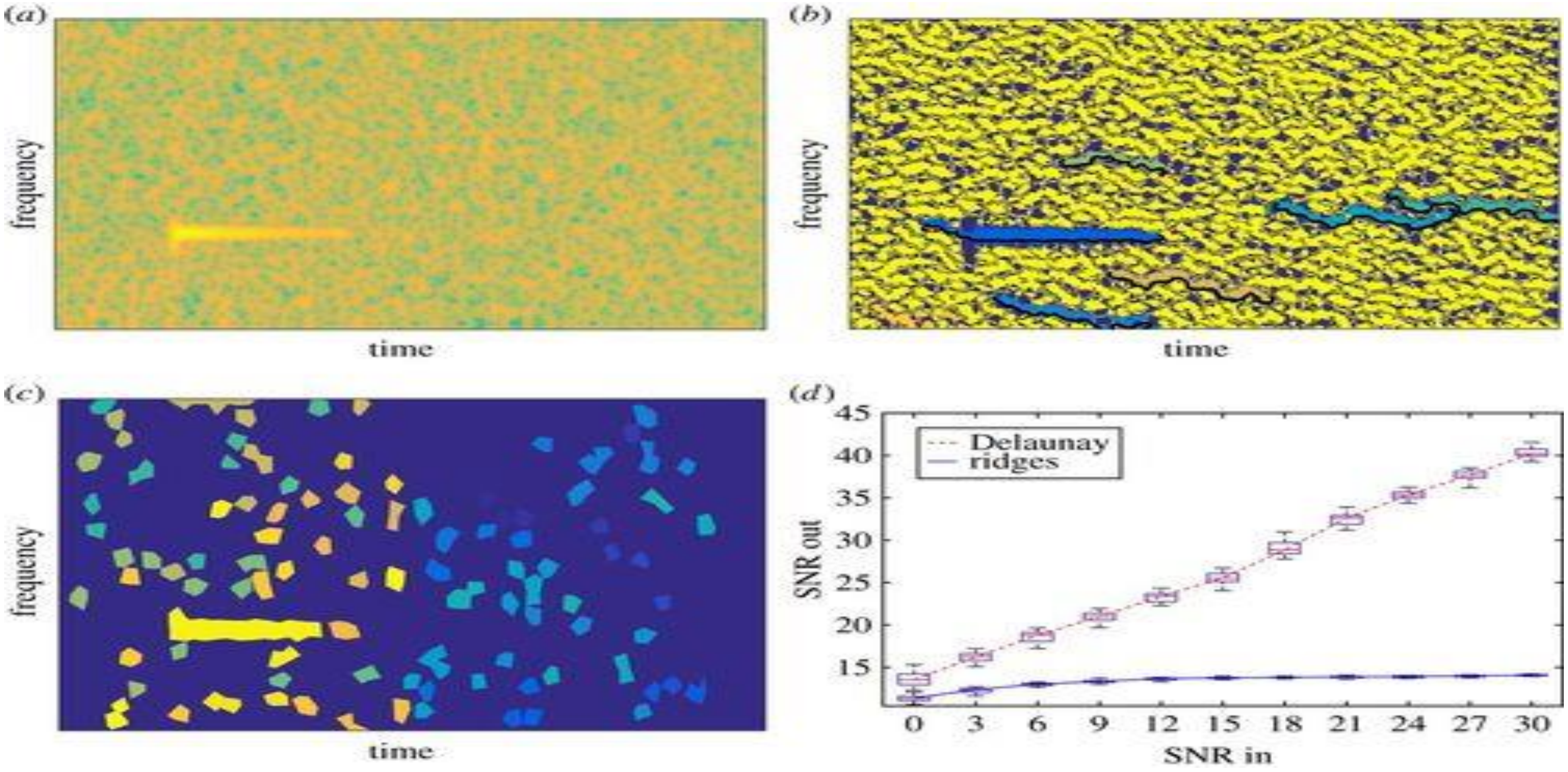
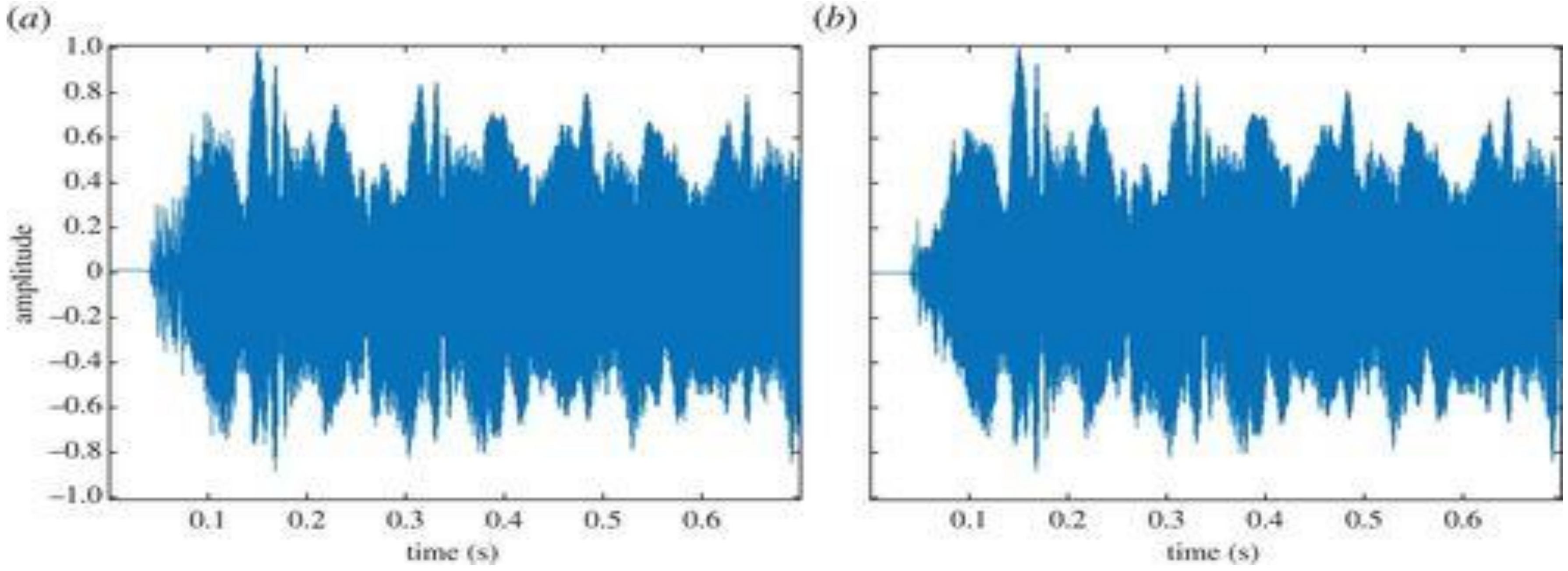


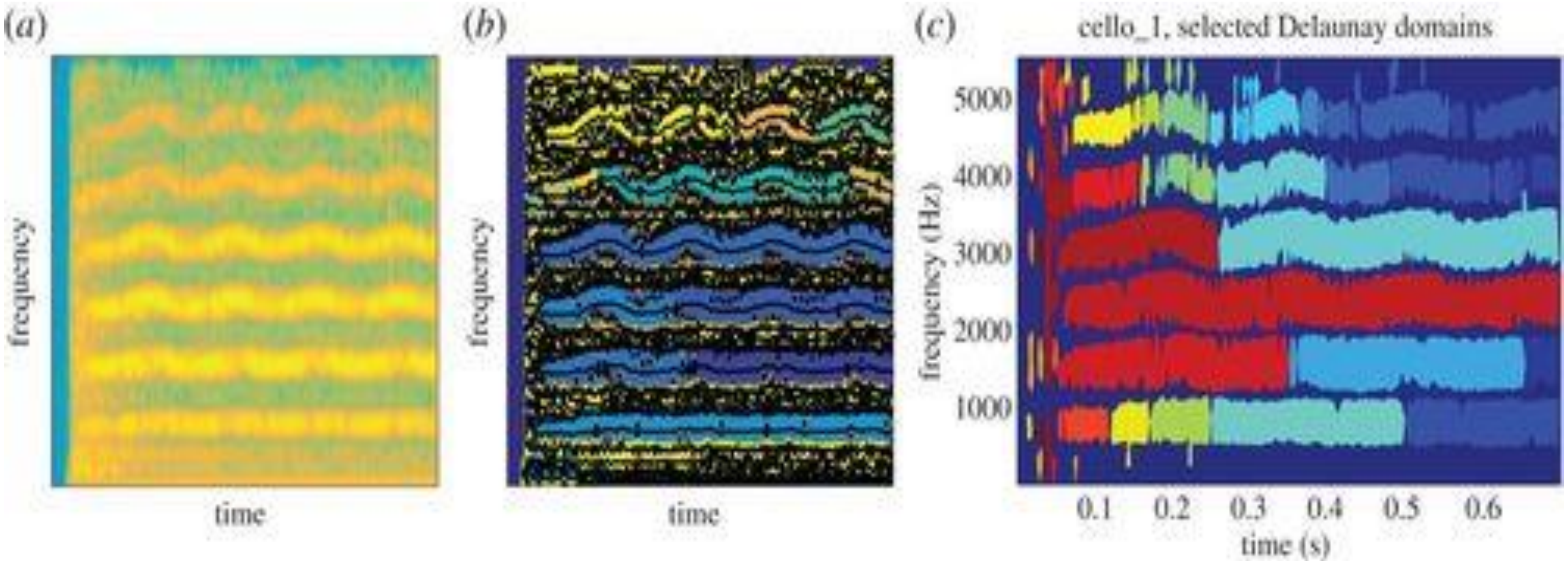
In this figure (a) Spectrogram of a noisy damped tone with a sharp attack (SNR= 0 dB). (b) Main ridges and basins of attraction derived from the RV. (c) Mode domains derived from Delaunay triangulation. (d) Denoising performance of the mode reconstruction methods.

Analysis of the sound of a Cello



(*a*) Cello sound, original signal; and (*b*) cello sound, reconstructed signal after mode domain extraction derived from Delaunay triangulation.

Analysis of the sound of a Cello



(*a*) Spectrogram of a cello sound. (*b*) Main ridges and basins of attraction derived from the RV. (*c*) Mode domains derived from Delaunay triangulation.

Summary

- Basic introduction to Time-Frequency
  - Understand your signal and what information you are seeking to extract
- Reassignment Methods
  - Useful tools in dealing with multicomponent signals

References

See below and references therein

Adaptive multimode signal reconstruction from time–frequency representations

Sylvain Meignen, Thomas Oberlin, Philippe Depalle, Patrick Flandrin and Stephen McLaughlin

Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering SciencesVolume 374, Issue 2065

https://doi.org/10.1098/rsta.2015.0205

Note

Delaunay- based software is available at

http://perso.ens-lyon.fr/patrick.flandrin/zeros.html

Plus see

http://www.ens-lyon.fr/PHYSIQUE/Equipe3/ANR_ASTRES/index2.html

Ridge-based Matlab code is available at

http://oberlin.perso.enseeiht.fr/files/mcs_ridge_decomposition.zip