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Cramér-Rao Bound for a Mixture of Real- and Integer-valued Parameter Vectors and its Application to the Linear Regression Model

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Abstract

Performance lower bounds are known to be a fundamental design tool in parametric estimation theory. A plethora of deterministic bounds exist in the literature, ranging from the general Barankin bound to the well-known Cramér-Rao bound (CRB), the latter providing the optimal mean square error performance of locally unbiased estimators. In this contribution, we are interested in the estimation of mixed real- and integer-valued parameter vectors. We propose a closed-form lower bound expression leveraging on the general CRB formulation, being the limiting form of the McAulay-Seidman bound. Such formulation is the key point to take into account integer-valued parameters. As a particular case of the general form, we provide closed-form expressions for the Gaussian observation model. One noteworthy point is the assessment of the asymptotic efficiency of the maximum likelihood estimator for a linear regression model with mixed parameter vectors and known noise covariance matrix, thus complementing the rather rich literature on that topic. A representative carrier-phase based precise positioning example is provided to support the discussion and show the usefulness of the proposed lower bound.

Keywords: Cramér-Rao bound, McAulay-Seidman bound, mixed real-integer parameter vector estimation, linear regression, GNSS, ambiguity resolution.

1. Introduction

Integer parameter estimation appears in many signal processing, biology and communications problems, to name a few. For instance, consider a multi-hypothesis testing problem where we want to identify the received signal over a (finite) set of possible transmitted signals, then a solution is to maximize the log-likelihood function over the (integer) set of candidates. Another problem involving estimation of integer quantities, jointly with a real-valued vector, is that of carrier-phase based precise positioning in the context of Global Navigation Satellite Systems (GNSS) receivers. In the geodesy and navigation community, a well known estimation approach is referred to as Real Time Kinematic (RTK) positioning [1]. Carrier-phase measurements have an unknown integer part, referred to as the *ambiguity*, to be estimated in order to achieve cm-level accuracy on the real-valued unknown position of the receiver. The framework that underpins precise GNSS carrier phase-based ambiguity resolution is the theory of integer aperture estimation [2][3], which also applies to other carrier phase-based interferometric techniques, such as Very Long Baseline Interferometry (VLBI) [4], Interferometric Synthetic Aperture Radar (InSAR) [5], or underwater acoustic carrier phase-based positioning [6].

Regardless of the estimation problem addressed, when designing and assessing estimators it is of fundamental importance to know the minimum achievable performance, that is, to obtain tight performance lower bounds (LBs). In general, in estimation problems we are interested in minimal performance bounds in the mean squared error (MSE) sense, which provide the best achievable performance on the estimation of parameters of a signal corrupted by noise. There are two main categories of LBs, *deterministic* and *Bayesian* [7]. While the former considers that the parameters to be estimated are deterministic and evaluate the *locally best* estimator performance, the latter consider random parameters with a given *a priori* probability and evaluate the *globally best* estimator behavior. In this contribution we are interested in deterministic parameter estimation, thus only the first class will be discussed.

It is worth saying that such LBs have been proved to be extremely useful, not only for characterizing an estimator asymptotic performance, but also for system design [7, 8, 9]. The most popular LB is the well-known Cramér-Rao Bound (CRB) derived for real-valued parameter vector, mainly due to: *i*) its simplicity of calculation,

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for instance using the Slepian-Bangs' formula [10]; *ii*) it is the lowest bound on the MSE of any unbiased estimator (i.e., it considers local unbiasedness at the vicinity of any selected parameters' value); and *iii*) it is asymptotically attained by maximum likelihood estimators (MLEs) under certain conditions (i.e., high signal-to-noise ratio (SNR) [11] and/or large number of snapshots [12]), that is, MLEs are asymptotically efficient. Inherent limitations of such CRBs are their inability to: predict the *threshold phenomena*; provide tight bounds in certain cases [13]; and deal with integer-valued parameter estimation, which is the contribution of this article.

Since the seminal CRB works, several deterministic bounds have been proposed in the literature [14, 15, 16, 17, 18, 19, 20, 21, 22] to provide computable approximations of the Barankin bound (BB) [23], which is the tightest (greatest) LB for any absolute moment of order greater than 1 of unbiased estimators. In fact, the BB considers *uniform unbiasedness* (i.e., unbiasedness over an interval of parameter values including the selected value), resulting in a much stronger restriction than the local unbiasedness condition of the CRB, but not admitting an analytic solution in general.

In this contribution, in order to obtain a CRB-like closed-form expression for the estimation of mixed parameter vectors, including both real- and integer-valued parameters, we leverage on the McAulay-Seidman bound (MSB) [16]. The MSB is the BB approximation obtained from a discretization of the Barankin uniform unbiasedness constraint, using a set of selected values of the parameter vector, so-called *test points*. The MSB yields to a general definition of the CRB, expressed as a limiting form of the MSB. We derive a closed-form general CRB expression for mixed parameter vectors and provide its particular closed-form for the Gaussian observation model, which encompasses the well known conditional and unconditional observation models [24].

On another note, one must keep in mind that in many problems of practical interest, including the general (nonlinear) case of the problem under consideration, no evidence of the achievability of a given LB by realizable estimators exists [7, 8, 13]. Thus, from a practical perspective, the LB considered may be too optimistic and unable to represent the actual performance of any estimator. To circumvent the unavailability of LB achievability results, a solution relates to the derivation of an upper bound to provide a complementary vision to that of the LB. Unfortunately, upper bounds on the MSE of unbiased estimates do not generally exist if the observation space is unbounded. Nonetheless, upper bounds on the statistical performance (not necessarily the MSE) may exist for specific estimators (not necessarily unbiased) in specific estimation problems [1, 25, 26, 27]. In particular, for the mixed integer linear regression model, a rich literature on the statistical performances of various estimators is already available (see [1] and references therein), and an upper bound on the probability that the MLE of the real-valued parameter vector lies in a certain region exists [25].

The article is organized as follows: Section 2 provides background on deterministic LBs and their derivation as a norm minimization problem, mainly focused on the CRB as the limiting form of the MSB. Section 3 details the derivation of the new bound, in the general case and for the Gaussian observation model. It establishes the asymptotic efficiency of the MLE for a linear regression model with mixed parameter vectors and known noise covariance matrix, and sketches possible generalizations and outlooks. These results are then particularized for a linear regression problem, serving as motivating example and discussed in Section 4. The paper concludes with a discussion of the results in Section 5.

2. Background on McAulay-Seidman and Cramér-Rao Bounds for a Real-valued Parameter Vector

2.1. The McAulay-Seidman Bound

Let $\mathbf{y}$[1] be a random real-valued observations vector and $\Omega \subset \mathbb{R}^M$ the observation space. Denote by $p(\mathbf{y};\boldsymbol{\theta}) \triangleq p(\mathbf{y}|\boldsymbol{\theta})$ the pdf of the observations conditional on an unknown deterministic real-valued parameter vector $\boldsymbol{\theta} \in \Theta \subset \mathbb{R}^K$. Let $\mathcal{L}_2(\Omega)$ be the real vector space of square integrable functions over Ω. If we consider an estimator $\widehat{\mathbf{g}(\boldsymbol{\theta}^0)}(\mathbf{y}) \in \mathcal{L}_2^N(\Omega)$ of $\mathbf{g}(\boldsymbol{\theta}^0)$, where $\boldsymbol{\theta}^0$ is a selected value of the parameter $\boldsymbol{\theta}$ and $\mathbf{g}(\boldsymbol{\theta}) = (g_1(\boldsymbol{\theta}),\ldots,g_N(\boldsymbol{\theta}))^\top$ is a real-valued function vector, then the MSE matrix writes,

$$\mathbf{MSE}_{\boldsymbol{\theta}^0}\left(\widehat{\mathbf{g}(\boldsymbol{\theta}^0)}\right) = \mathbb{E}_{\mathbf{y};\boldsymbol{\theta}^0}\left[\left(\widehat{\mathbf{g}(\boldsymbol{\theta}^0)}(\mathbf{y}) - \mathbf{g}(\boldsymbol{\theta}^0)\right)(\cdot)^\top\right]. \tag{1}$$

By noticing that (1) is a Gram matrix associated with the scalar product $\langle h(\mathbf{y})|l(\mathbf{y})\rangle_{\boldsymbol{\theta}^0} = \mathbb{E}_{\mathbf{y};\boldsymbol{\theta}^0}[h(\mathbf{y})l(\mathbf{y})]$, the search for a LB on the MSE (1) (w.r.t. the Löwner ordering for positive symmetric matrices [28]) can be performed with two equivalent fundamental results: the generalization of the Cauchy-Schwartz inequality to Gram matrices (generally referred to as the "covariance inequality"

[1]Italic indicates a scalar quantity, as in a; lower case boldface indicates a column vector quantity, as in $\mathbf{a}$; upper case boldface indicates a matrix quantity, as in $\mathbf{A}$. The n-th row and m-th column element of the matrix $\mathbf{A}$ will be denoted by $A_{n,m}$ or $[\mathbf{A}]_{n,m}$. The n-th coordinate of the column vector $\mathbf{a}$ will be denoted by a_n or $[\mathbf{a}]_n$. The matrix/vector transpose is indicated by a superscript $(\cdot)^\top$ as in $\mathbf{A}^\top$. $|\mathbf{A}|$ is the determinant of the square matrix $\mathbf{A}$. $[\mathbf{A}\ \mathbf{B}]$ and $\begin{bmatrix}\mathbf{A}\\\mathbf{B}\end{bmatrix}$ denote the matrix resulting from the horizontal and the vertical concatenation of $\mathbf{A}$ and $\mathbf{B}$, respectively. $\mathbf{I}_M$ is the identity matrix of dimension M. $\mathbf{1}_M$ is a M-dimensional vector with all components equal to one. For two matrices $\mathbf{A}$ and $\mathbf{B}$, $\mathbf{A} \geq \mathbf{B}$ means that $\mathbf{A}-\mathbf{B}$ is positive semi-definite (Löwner partial ordering). $\|\cdot\|$ denotes a norm. If $\boldsymbol{\theta} = [\theta_1,\theta_2,\ldots,\theta_K]^\top$, then: $\frac{\partial}{\partial\boldsymbol{\theta}} = \left[\frac{\partial}{\partial\theta_1},\frac{\partial}{\partial\theta_2},\ldots,\frac{\partial}{\partial\theta_K}\right]^\top$, $\frac{\partial}{\partial\boldsymbol{\theta}^\top} = \left[\frac{\partial}{\partial\theta_1},\frac{\partial}{\partial\theta_2},\ldots,\frac{\partial}{\partial\theta_K}\right]$ and $\frac{\partial\mathbf{h}(\boldsymbol{\theta}_0,\mathbf{y})}{\partial\boldsymbol{\theta}} = \left.\frac{\partial\mathbf{h}(\boldsymbol{\theta},\mathbf{y})}{\partial\boldsymbol{\theta}}\right|_{\boldsymbol{\theta}_0}$. $p(\mathbf{y};\boldsymbol{\theta}) \triangleq p(\mathbf{y}|\boldsymbol{\theta})$ denotes the probability density function (pdf) of $\mathbf{y}$ parameterized by $\boldsymbol{\theta}$. $\mathbb{E}_{\mathbf{y};\boldsymbol{\theta}}[\mathbf{g}(\mathbf{y})]$ denote the statistical expectation of the vector of functions $\mathbf{g}(\cdot)$ with respect to $\mathbf{y}$ parameterized by $\boldsymbol{\theta}$. For the sake of simplicity, $(\mathbf{g}(\mathbf{y}))(\cdot)^\top \triangleq \mathbf{g}(\mathbf{y})\mathbf{g}(\mathbf{y})^\top$.

[18]) and the minimization of a norm under linear constraints (LCs) [17, 19, 20]. We shall prefer the "norm minimization" form as its use requires explicitly the selection of appropriate constraints, which then determine the value of the LB on the MSE matrix, hence providing a clear understanding of the hypotheses associated with the different LBs on the MSE. To avoid the trivial solution $\widehat{\mathbf{g}(\boldsymbol{\theta}^0)}(\mathbf{y}) = \mathbf{g}(\boldsymbol{\theta}^0)$, some constraints must be added. In that perspective, Barankin [23] introduced the *uniform unbiasedness* formulation,

$$\mathbb{E}_{\mathbf{y};\boldsymbol{\theta}}\left[\widehat{\mathbf{g}(\boldsymbol{\theta}^0)}(\mathbf{y})\right] = \mathbf{g}(\boldsymbol{\theta}), \forall \boldsymbol{\theta} \in \Theta, \tag{2a}$$

leading to the Barankin bound (BB),

$$\begin{aligned} &\min_{\widehat{\mathbf{g}(\boldsymbol{\theta}^0)} \in \mathcal{L}_2^N(\Omega)} \left\{ \mathbf{MSE}_{\boldsymbol{\theta}^0}\left(\widehat{\mathbf{g}(\boldsymbol{\theta}^0)}\right)\right\} \\ &\text{s.t. } \mathbb{E}_{\mathbf{y};\boldsymbol{\theta}}\left[\widehat{\mathbf{g}(\boldsymbol{\theta}^0)}(\mathbf{y})\right] = \mathbf{g}(\boldsymbol{\theta}), \forall \boldsymbol{\theta} \in \Theta, \end{aligned} \tag{2b}$$

which does not admit an analytic solution in general. The McAulay-Seidman bound (MSB) is the computable BB approximation obtained from a discretization of the *uniform unbiasedness* constraint (2a). Let $\{\boldsymbol{\theta}\}^L \triangleq \{\boldsymbol{\theta}\}^{[1,L]} = \left\{\boldsymbol{\theta}^1, \ldots, \boldsymbol{\theta}^L\right\} \in \Theta^L$ be a subset of L selected values of $\boldsymbol{\theta}$ (a.k.a. *test points*). Then, any unbiased estimator $\widehat{\mathbf{g}(\boldsymbol{\theta}^0)}(\mathbf{y})$ verifying (2a) must comply with the following subset of L LCs,

$$\mathbb{E}_{\mathbf{y};\boldsymbol{\theta}^l}\left[\widehat{\mathbf{g}(\boldsymbol{\theta}^0)}(\mathbf{y})\right] = \mathbf{g}\left(\boldsymbol{\theta}^l\right), \quad 1 \le l \le L, \tag{3a}$$

which can be recast as

$$\underbrace{\mathbb{E}_{\mathbf{y};\boldsymbol{\theta}^0}\left[\boldsymbol{\upsilon}_{\boldsymbol{\theta}^0}\left(\mathbf{y};\{\boldsymbol{\theta}\}^L\right)\left(\widehat{\mathbf{g}(\boldsymbol{\theta}^0)}(\mathbf{y}) - \mathbf{g}(\boldsymbol{\theta}^0)\right)^\top\right]}_{\boldsymbol{\Gamma}} = \underbrace{\begin{bmatrix} \left(\mathbf{g}(\boldsymbol{\theta}^1) - \mathbf{g}(\boldsymbol{\theta}^0)\right)^\top \\ \vdots \\ \left(\mathbf{g}\left(\boldsymbol{\theta}^L\right) - \mathbf{g}(\boldsymbol{\theta}^0)\right)^\top \end{bmatrix}}_{\mathbf{V}}, \tag{3b}$$

where $\boldsymbol{\upsilon}_{\boldsymbol{\theta}^0}\left(\mathbf{y};\{\boldsymbol{\theta}\}^L\right) = \left(\upsilon_{\boldsymbol{\theta}^0}(\mathbf{y};\boldsymbol{\theta}^1), \ldots, \upsilon_{\boldsymbol{\theta}^0}\left(\mathbf{y};\boldsymbol{\theta}^L\right)\right)^\top$, $\upsilon_{\boldsymbol{\theta}^0}(\mathbf{y};\boldsymbol{\theta}) = p(\mathbf{y};\boldsymbol{\theta})/p(\mathbf{y};\boldsymbol{\theta}^0)$, is the vector of likelihood ratios associated to $\{\boldsymbol{\theta}\}^L$. The L LCs (3b) yields the approximation of (2b) proposed by McAulay and Seidman [20],

$$\min_{\widehat{\mathbf{g}(\boldsymbol{\theta}^0)} \in \mathcal{L}_2^N(\Omega)} \left\{ \mathbf{MSE}_{\boldsymbol{\theta}^0}\left(\widehat{\mathbf{g}(\boldsymbol{\theta}^0)}\right)\right\} \text{ s.t. } \boldsymbol{\Gamma} = \mathbf{V}, \tag{4a}$$

and defines the MSB (Lemma 1 in [29]) [16, 20]

$$\begin{aligned} &\mathbb{E}_{\mathbf{y};\boldsymbol{\theta}^0}\left[\left(\widehat{\mathbf{g}(\boldsymbol{\theta}^0)}(\mathbf{y}) - \mathbf{g}(\boldsymbol{\theta}^0)\right)(\cdot)^\top\right] \ge \boldsymbol{\Delta}_{\mathbf{g}}(\boldsymbol{\theta}^0)\,\mathbf{R}_{\boldsymbol{\upsilon}_{\boldsymbol{\theta}^0}}^{-1}\boldsymbol{\Delta}_{\mathbf{g}}^\top(\boldsymbol{\theta}^0), \\ &\boldsymbol{\Delta}_{\mathbf{g}}(\boldsymbol{\theta}^0) \triangleq \boldsymbol{\Delta}_{\mathbf{g}}\left(\boldsymbol{\theta}^0, \{\boldsymbol{\theta}\}^L\right) \\ &= \left[\begin{array}{ccc} \mathbf{g}(\boldsymbol{\theta}^1) - \mathbf{g}(\boldsymbol{\theta}^0) & \ldots & \mathbf{g}\left(\boldsymbol{\theta}^L\right) - \mathbf{g}(\boldsymbol{\theta}^0) \end{array}\right], \\ &\mathbf{R}_{\boldsymbol{\upsilon}_{\boldsymbol{\theta}^0}} \triangleq \mathbf{R}_{\boldsymbol{\upsilon}_{\boldsymbol{\theta}^0}}\left(\{\boldsymbol{\theta}\}^L\right) = \mathbb{E}_{\mathbf{y};\boldsymbol{\theta}^0}\left[\boldsymbol{\upsilon}_{\boldsymbol{\theta}^0}\left(\mathbf{y};\{\boldsymbol{\theta}\}^L\right)\boldsymbol{\upsilon}_{\boldsymbol{\theta}^0}^\top\left(\mathbf{y};\{\boldsymbol{\theta}\}^L\right)\right], \end{aligned} \tag{4b}$$

a generalization of the Hammersley-Chapman-Robbins bound (HaChRB) previously introduced in [15, 30] for 2 test points ($L = 2$).

2.2. CRB as a Limiting Form of the MSB

The CRB can be defined for any absolute moment (greater than 1) as the limiting form of the HaChRB [15, 30], as showed in [23]. The extension to the multidimensional real-valued parameters case for the MSE (i.e., absolute moment of order 2) was introduced in [16], allowing to define the CRB as the limiting form of the MSB (4b). Considering the subset of test points

$$\begin{aligned} \{\boldsymbol{\theta}\}^{1+K} &= \left\{\boldsymbol{\theta}^0, \boldsymbol{\theta}^0 + \mathbf{i}_1 d\theta_1, \ldots, \boldsymbol{\theta}^0 + \mathbf{i}_K d\theta_K\right\} \\ &\text{under } d\theta_k \neq 0, \quad 1 \le k \le K, \end{aligned}$$

where $\mathbf{i}_k$ is the kth column of the identity matrix $\mathbf{I}_K$, leads to

$$\upsilon_{\boldsymbol{\theta}^0}\left(\mathbf{y};\{\boldsymbol{\theta}\}^{1+K}\right) = \left[\begin{array}{cccc} 1 & \frac{p(\mathbf{y};\boldsymbol{\theta}^0 + \mathbf{i}_1 d\theta_1)}{p(\mathbf{y};\boldsymbol{\theta}^0)} & \ldots & \frac{p(\mathbf{y};\boldsymbol{\theta}^0 + \mathbf{i}_K d\theta_K)}{p(\mathbf{y};\boldsymbol{\theta}^0)} \end{array}\right]^\top,$$

$$\Delta_{\mathbf{g}}(\boldsymbol{\theta}^0) = \left[\mathbf{0} \quad \mathbf{g}(\boldsymbol{\theta}^0 + \mathbf{i}_1 d\theta_1) - \mathbf{g}(\boldsymbol{\theta}^0) \quad \ldots \quad \mathbf{g}(\boldsymbol{\theta}^0 + \mathbf{i}_K d\theta_K) - \mathbf{g}(\boldsymbol{\theta}^0)\right]$$

and, with $d\boldsymbol{\theta} = (d\theta_1, \ldots, d\theta_K)^\top$, yields to (see Appendix A)

$$\Delta_{\mathbf{g}}(\boldsymbol{\theta}^0)\,\mathbf{R}_{\upsilon_{\boldsymbol{\theta}^0}}^{-1}\boldsymbol{\Delta}_{\mathbf{g}}^\top(\boldsymbol{\theta}^0) = \boldsymbol{\Lambda}_{\mathbf{g}}(\boldsymbol{\theta}^0, d\boldsymbol{\theta})\,\widetilde{\mathbf{F}}(\boldsymbol{\theta}^0, d\boldsymbol{\theta})^{-1}\boldsymbol{\Lambda}_{\mathbf{g}}^\top(\boldsymbol{\theta}^0, d\boldsymbol{\theta}), \tag{5a}$$

$$\widetilde{\mathbf{F}}(\boldsymbol{\theta}^0, d\boldsymbol{\theta}) = \mathbb{E}_{\mathbf{y};\boldsymbol{\theta}^0}\left[\left(\begin{array}{c} \frac{p(\mathbf{y};\boldsymbol{\theta}^0 + \mathbf{i}_1 d\theta_1) - p(\mathbf{y};\boldsymbol{\theta}^0)}{d\theta_1 p(\mathbf{y};\boldsymbol{\theta}^0)} \\ \vdots \\ \frac{p(\mathbf{y};\boldsymbol{\theta}^0 + \mathbf{i}_K d\theta_K) - p(\mathbf{y};\boldsymbol{\theta}^0)}{d\theta_K p(\mathbf{y};\boldsymbol{\theta}^0)} \end{array}\right)(.)^\top\right] \tag{5b}$$

$$\Lambda_{\mathbf{g}}(\boldsymbol{\theta}^0, d\boldsymbol{\theta}) = \left[\begin{array}{ccc} \frac{\mathbf{g}(\boldsymbol{\theta}^0 + \mathbf{i}_1 d\theta_1) - \mathbf{g}(\boldsymbol{\theta}^0)}{d\theta_1} & \ldots & \frac{\mathbf{g}(\boldsymbol{\theta}^0 + \mathbf{i}_K d\theta_K) - \mathbf{g}(\boldsymbol{\theta}^0)}{d\theta_K} \end{array}\right], \tag{5c}$$

which results in a general definition of the $\mathbf{CRB}_{\mathbf{g}|\boldsymbol{\theta}}(\boldsymbol{\theta}^0)$ as

$$\text{CRB}_{\mathbf{g}|\boldsymbol{\theta}}(\boldsymbol{\theta}^0) = \lim_{\sup\{d\theta_1 \neq 0, \ldots, d\theta_K \neq 0\} \to 0} \boldsymbol{\Lambda}_{\mathbf{g}}(\boldsymbol{\theta}^0, d\boldsymbol{\theta})\,\widetilde{\mathbf{F}}(\boldsymbol{\theta}^0, d\boldsymbol{\theta})^{-1}\boldsymbol{\Lambda}_{\mathbf{g}}^\top(\boldsymbol{\theta}^0, d\boldsymbol{\theta}). \tag{6a}$$

If $\boldsymbol{\theta}^0 \in \Theta \subset \mathbb{R}^K$ and $\mathbf{g}(\boldsymbol{\theta})$ and $p(\mathbf{y};\boldsymbol{\theta})$ are C^1 at $\boldsymbol{\theta}^0$, then (6a) yields the well known Fisher Information Matrix

(FIM) $\mathbf{F}(\boldsymbol{\theta})$ and the usual CRB expression

$$\mathbf{F}(\boldsymbol{\theta}) = \mathbb{E}_{\mathbf{y};\boldsymbol{\theta}}\left[\frac{\partial \ln p(\mathbf{y};\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}(.)^{\top}\right], \tag{6b}$$

$$\mathbf{CRB}_{\mathbf{g}|\boldsymbol{\theta}}\left(\boldsymbol{\theta}^0\right) = \frac{\partial \mathbf{g}\left(\boldsymbol{\theta}^0\right)}{\partial \boldsymbol{\theta}^{\top}} \mathbf{F}\left(\boldsymbol{\theta}^0\right)^{-1} \left(\frac{\partial \mathbf{g}\left(\boldsymbol{\theta}^0\right)}{\partial \boldsymbol{\theta}^{\top}}\right)^{\top}. \tag{6c}$$

3. CRB for a Mixture of Real-valued and Integer-valued Parameters

Leveraging the MSB and CRB results presented in the previous Section 2, we derive in this section a LB for deterministic parameter vector estimation, where such vector contains both real-valued and integer-valued parameters. A general result is provided, then particularized for the case of Gaussian observations.

3.1. General CRB for Mixed Parameter Vectors

The main result derived in this article is summarized in the form of Theorem 1. A corollary follows, which simplifies the former in a particular class of models.

Theorem 1 (General CRB for mixed parameter vectors)**.** *Assume a set of observations $\mathbf{y} \in \Omega \subset \mathbb{R}^M$ and an unknown deterministic real-valued parameter vector $\boldsymbol{\theta} \in \Theta \subset \mathbb{R}^K$ where $\boldsymbol{\theta}^{\top} = \left[\boldsymbol{\omega}^{\top}, \mathbf{z}^{\top}\right]$, $\boldsymbol{\omega} \in \mathbb{R}^{K_\omega}$, $\mathbf{z} \in \mathbb{Z}^{K_z}$, $K_\omega + K_z = K$. Those quantities are related through a statistical model of the form $\mathbf{y}|\boldsymbol{\theta} \sim p(\mathbf{y}|\boldsymbol{\theta})$, which is available. Then, the MSE of any unbiased estimator of a function $\mathbf{g}\left(\boldsymbol{\theta}^0\right) \in \mathcal{L}_2(\Omega)$ for a selected value of the parameter $\boldsymbol{\theta}^0$ is lower bounded by*

$$\mathbf{CRB}_{\mathbf{g}|\boldsymbol{\theta}}\left(\boldsymbol{\theta}^0\right) = \boldsymbol{\Lambda}_{\mathbf{g}}\left(\boldsymbol{\theta}^0\right) \overline{\mathbf{F}}\left(\boldsymbol{\theta}^0\right)^{-1} \boldsymbol{\Lambda}_{\mathbf{g}}^{\top}\left(\boldsymbol{\theta}^0\right), \tag{7}$$

with

$$\Lambda_{\mathbf{g}}\left(\boldsymbol{\theta}^0\right) = \left[\begin{array}{cccc} \frac{\partial \mathbf{g}\left(\boldsymbol{\theta}^0\right)}{\partial \boldsymbol{\omega}^{\top}} & \mathbf{g}\left(\boldsymbol{\theta}^1\right) - \mathbf{g}\left(\boldsymbol{\theta}^0\right) & \ldots & \mathbf{g}\left(\boldsymbol{\theta}^{2K_z}\right) - \mathbf{g}\left(\boldsymbol{\theta}^0\right) \end{array}\right] \tag{8}$$

$$\overline{\mathbf{F}}\left(\boldsymbol{\theta}^0\right) = \left[\begin{array}{cc} \mathbf{F}_{\boldsymbol{\omega}|\boldsymbol{\theta}}\left(\boldsymbol{\theta}^0\right) & \mathbf{H}\left(\boldsymbol{\theta}^0\right) \\ \mathbf{H}\left(\boldsymbol{\theta}^0\right)^{\top} & \mathbf{MS}_{\mathbf{z}|\boldsymbol{\theta}}\left(\boldsymbol{\theta}^0\right) \end{array}\right], \tag{9}$$

where the test points $\{\boldsymbol{\theta}\}^{2K_z}$ are defined as

$$\boldsymbol{\theta}^j = \boldsymbol{\theta}^0 + (-1)^{j-1} \mathbf{i}_{K_\omega + \left\lfloor \frac{j+1}{2} \right\rfloor}, \quad 1 \leq j \leq 2K_z, \tag{10}$$

that is,

$$\left[\boldsymbol{\theta}^1, \boldsymbol{\theta}^2, \ldots, \boldsymbol{\theta}^{2K_z-1}, \boldsymbol{\theta}^{2K_z}\right] = \left[\boldsymbol{\theta}^0 + \mathbf{i}_{K_\omega+1}, \boldsymbol{\theta}^0 - \mathbf{i}_{K_\omega+1}, \ldots, \boldsymbol{\theta}^0 + \mathbf{i}_K, \boldsymbol{\theta}^0 - \mathbf{i}_K\right].$$

The different terms in $\overline{\mathbf{F}}\left(\boldsymbol{\theta}^0\right)$ are given by

$$\mathbf{F}_{\boldsymbol{\omega}|\boldsymbol{\theta}}\left(\boldsymbol{\theta}^0\right) = \mathbb{E}_{\mathbf{y};\boldsymbol{\theta}^0}\left[\frac{\partial \ln p\left(\mathbf{y};\boldsymbol{\theta}^0\right)}{\partial \boldsymbol{\omega}}(.)^{\top}\right], \tag{11a}$$

$$\mathbf{H}\left(\boldsymbol{\theta}^0\right) = \mathbb{E}_{\mathbf{y};\boldsymbol{\theta}^0}\left[\frac{\partial \ln p\left(\mathbf{y};\boldsymbol{\theta}^0\right)}{\partial \boldsymbol{\omega}} \mathbf{t}_{2K_z}^{\top}\right] \tag{11b}$$

$$= \left[\begin{array}{cccc} \mathbf{h}\left(\boldsymbol{\theta}^0, \boldsymbol{\theta}^1\right) & \mathbf{h}\left(\boldsymbol{\theta}^0, \boldsymbol{\theta}^2\right) & \ldots & \mathbf{h}\left(\boldsymbol{\theta}^0, \boldsymbol{\theta}^{2K_z}\right) \end{array}\right], \tag{11c}$$

$$\mathbf{MS}_{\mathbf{z}|\boldsymbol{\theta}}\left(\boldsymbol{\theta}^0\right) = \mathbb{E}_{\mathbf{y};\boldsymbol{\theta}^0}\left[\mathbf{t}_{2K_z} \mathbf{t}_{2K_z}^{\top}\right] - \mathbf{1}_{2K_z} \mathbf{1}_{2K_z}^{\top}, \tag{11d}$$

where $\mathbf{t}_{2K_z}$ is defined as

$$\mathbf{t}_{2K_z} \triangleq \boldsymbol{\upsilon}_{\boldsymbol{\theta}^0}\left(\mathbf{y}; \{\boldsymbol{\theta}\}^{2K_z}\right) \tag{11e}$$

$$= \left(\frac{p\left(\mathbf{y};\boldsymbol{\theta}^1\right)}{p\left(\mathbf{y};\boldsymbol{\theta}^0\right)}, \frac{p\left(\mathbf{y};\boldsymbol{\theta}^2\right)}{p\left(\mathbf{y};\boldsymbol{\theta}^0\right)}, \ldots, \frac{p\left(\mathbf{y};\boldsymbol{\theta}^{2K_z}\right)}{p\left(\mathbf{y};\boldsymbol{\theta}^0\right)}\right)^{\top}. \tag{11f}$$

Proof. First, notice that in the real-valued parameter case, that is, if $\theta_k^0 \in \mathbb{R}$, and both $\mathbf{g}(\boldsymbol{\theta})$ and $p(\mathbf{y};\boldsymbol{\theta})$ are C^1 at θ_k^0, then, the constraints associated to the following two test points, $\left\{\boldsymbol{\theta}^0 + \mathbf{i}_k d\theta_k, \boldsymbol{\theta}^0 + \mathbf{i}_k(-d\theta_k)\right\} = \left\{\boldsymbol{\theta}^0 + \mathbf{i}_k d\theta_k, \boldsymbol{\theta}^0 - \mathbf{i}_k d\theta_k\right\}$,

$$\mathbb{E}_{\mathbf{y};\boldsymbol{\theta}^0}\left[\left(\begin{array}{c} \frac{p\left(\mathbf{y};\boldsymbol{\theta}^0+\mathbf{i}_k d\theta_k\right) - p\left(\mathbf{y};\boldsymbol{\theta}^0\right)}{d\theta_k p\left(\mathbf{y};\boldsymbol{\theta}^0\right)} \\ \frac{p\left(\mathbf{y};\boldsymbol{\theta}^0-\mathbf{i}_k d\theta_k\right) - p\left(\mathbf{y};\boldsymbol{\theta}^0\right)}{(-d\theta_k) p\left(\mathbf{y};\boldsymbol{\theta}^0\right)} \end{array}\right)\left(\widehat{\mathbf{g}\left(\boldsymbol{\theta}^0\right)}(\mathbf{y}) - \mathbf{g}\left(\boldsymbol{\theta}^0\right)\right)^{\top}\right] = \left[\begin{array}{c} \left(\frac{\mathbf{g}\left(\boldsymbol{\theta}^0+\mathbf{i}_k d\theta_k\right) - \mathbf{g}\left(\boldsymbol{\theta}^0\right)}{d\theta_k}\right)^{\top} \\ \left(\frac{\mathbf{g}\left(\boldsymbol{\theta}^0-\mathbf{i}_k d\theta_k\right) - \mathbf{g}\left(\boldsymbol{\theta}^0\right)}{(-d\theta_k)}\right)^{\top} \end{array}\right] \tag{12}$$

aim at the same single constraint in the limiting case where $d\theta_k \to 0$, $d\theta_k \neq 0$,

$$\mathbb{E}_{\mathbf{y};\boldsymbol{\theta}^0}\left[\frac{\partial \ln p\left(\mathbf{y};\boldsymbol{\theta}^0\right)}{\partial \theta_k}\left(\widehat{\mathbf{g}\left(\boldsymbol{\theta}^0\right)}(\mathbf{y}) - \mathbf{g}\left(\boldsymbol{\theta}^0\right)\right)^{\top}\right] = \frac{\partial \mathbf{g}\left(\boldsymbol{\theta}^0\right)}{\partial \theta_k}^{\top}. \tag{13}$$

However, this phenomenon is *unlikely* to happen if $\theta_k^0 \in \mathbb{Z}$ in the limiting case where $d\theta_k \to 0$, $d\theta_k \neq 0$, since (12) then becomes

$$\mathbb{E}_{\mathbf{y};\boldsymbol{\theta}^0}\left[\left(\begin{array}{c} \frac{p\left(\mathbf{y};\boldsymbol{\theta}^0+\mathbf{i}_k\right) - p\left(\mathbf{y};\boldsymbol{\theta}^0\right)}{p\left(\mathbf{y};\boldsymbol{\theta}^0\right)} \\ \frac{p\left(\mathbf{y};\boldsymbol{\theta}^0-\mathbf{i}_k\right) - p\left(\mathbf{y};\boldsymbol{\theta}^0\right)}{p\left(\mathbf{y};\boldsymbol{\theta}^0\right)} \end{array}\right)\left(\widehat{\mathbf{g}\left(\boldsymbol{\theta}^0\right)}(\mathbf{y}) - \mathbf{g}\left(\boldsymbol{\theta}^0\right)\right)^{\top}\right] = \left[\begin{array}{c} \left(\mathbf{g}\left(\boldsymbol{\theta}^0+\mathbf{i}_k\right) - \mathbf{g}\left(\boldsymbol{\theta}^0\right)\right)^{\top} \\ \left(\mathbf{g}\left(\boldsymbol{\theta}^0-\mathbf{i}_k\right) - \mathbf{g}\left(\boldsymbol{\theta}^0\right)\right)^{\top} \end{array}\right], \tag{14}$$

where $\left(p\left(\mathbf{y};\boldsymbol{\theta}^0+\mathbf{i}_k\right) - p\left(\mathbf{y};\boldsymbol{\theta}^0\right)\right)/p\left(\mathbf{y};\boldsymbol{\theta}^0\right)$ and $\left(p\left(\mathbf{y};\boldsymbol{\theta}^0-\mathbf{i}_k\right) - p\left(\mathbf{y};\boldsymbol{\theta}^0\right)\right)/p\left(\mathbf{y};\boldsymbol{\theta}^0\right)$ are unlikely to be linearly dependent (i.e., notice that $\widetilde{\mathbf{F}}\left(\boldsymbol{\theta}^0, d\boldsymbol{\theta}\right)$ in (5b) must be invertible to compute the $\mathbf{CRB}_{\mathbf{g}|\boldsymbol{\theta}}\left(\boldsymbol{\theta}^0\right)$ in (6a)). Therefore, *in most cases*, the combination of LCs (13) and

(14) yields, from Lemma 1 in [29], the general definition (7) of $\mathbf{CRB}_{\mathbf{g}|\boldsymbol{\theta}}\left(\boldsymbol{\theta}^0\right)$ where the different terms in $\overline{\mathbf{F}}\left(\boldsymbol{\theta}^0\right)$ are given by

$$\mathbf{F}_{\boldsymbol{\omega}|\boldsymbol{\theta}}\left(\boldsymbol{\theta}^0\right) = \mathbb{E}_{\mathbf{y};\boldsymbol{\theta}^0}\left[\frac{\partial \ln p\left(\mathbf{y};\boldsymbol{\theta}^0\right)}{\partial \boldsymbol{\omega}}\left(.\right)^\top\right], \tag{15a}$$

$$\mathbf{H}\left(\boldsymbol{\theta}^0\right) = \mathbb{E}_{\mathbf{y};\boldsymbol{\theta}^0}\left[\frac{\partial \ln p\left(\mathbf{y};\boldsymbol{\theta}^0\right)}{\partial \boldsymbol{\omega}}\left(\mathbf{t}_{2K_z} - \mathbf{1}_{2K_z}\right)^\top\right], \tag{15b}$$

$$\mathbf{MS}_{\mathbf{z}|\boldsymbol{\theta}}\left(\boldsymbol{\theta}^0\right) = \mathbb{E}_{\mathbf{y};\boldsymbol{\theta}^0}\left[\left(\mathbf{t}_{2K_z} - \mathbf{1}_{2K_z}\right)\left(\mathbf{t}_{2K_z} - \mathbf{1}_{2K_z}\right)^\top\right], \tag{15c}$$

and where $\mathbf{H}\left(\boldsymbol{\theta}^0\right)$ and $\mathbf{MS}_{\mathbf{z}|\boldsymbol{\theta}}\left(\boldsymbol{\theta}^0\right)$ can also been expressed as (11b) and (11d). □

Corollary 1. *If $\mathbf{g}(\boldsymbol{\theta}) = \boldsymbol{\theta}$, matrix $\boldsymbol{\Lambda}_{\boldsymbol{\theta}}\left(\boldsymbol{\theta}^0\right)$ in Theorem 1 simplifies to*

$$\boldsymbol{\Lambda}_{\boldsymbol{\theta}}\left(\boldsymbol{\theta}^0\right) = \begin{bmatrix}\mathbf{i}_1 & \dots & \mathbf{i}_{K_\omega} & \mathbf{i}_{K_\omega+1} & -\mathbf{i}_{K_\omega+1} & \dots & \mathbf{i}_K & -\mathbf{i}_K\end{bmatrix} \underset{(K_z=3)}{=} \begin{bmatrix}\mathbf{I}_{K_\omega} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & 1 & -1 & 0 & 0 & 0 & 0\\ \mathbf{0} & 0 & 0 & 1 & -1 & 0 & 0\\ \mathbf{0} & 0 & 0 & 0 & 0 & 1 & -1\end{bmatrix}. \tag{16}$$

3.2. The Gaussian Observation Model

Let us consider an M-dimensional Gaussian real vector $\mathbf{y}$ with mean $\mathbf{m}_{\mathbf{y}} = \mathbf{m}(\boldsymbol{\theta})$ and covariance matrix $\mathbf{C}_{\mathbf{y}} = \mathbf{C}(\boldsymbol{\theta})$: $\mathbf{y} \sim \mathcal{N}_M\left(\mathbf{m}(\boldsymbol{\theta}), \mathbf{C}(\boldsymbol{\theta})\right)$ and $p(\mathbf{y};\boldsymbol{\theta}) = p\left(\mathbf{y};\mathbf{m}(\boldsymbol{\theta}), \mathbf{C}(\boldsymbol{\theta})\right)$ such that

$$p\left(\mathbf{y};\mathbf{m}(\boldsymbol{\theta}), \mathbf{C}(\boldsymbol{\theta})\right) = \frac{e^{-\frac{1}{2}\left(\mathbf{y}-\mathbf{m}(\boldsymbol{\theta})\right)^\top \mathbf{C}^{-1}(\boldsymbol{\theta})\left(\mathbf{y}-\mathbf{m}(\boldsymbol{\theta})\right)}}{\sqrt{2\pi}^M\sqrt{\left|\mathbf{C}(\boldsymbol{\theta})\right|}}. \tag{17}$$

If we define

$$\mathbf{C}^{ij} = \left[\mathbf{C}\left(\boldsymbol{\theta}^i\right)^{-1} + \mathbf{C}\left(\boldsymbol{\theta}^j\right)^{-1} - \mathbf{C}\left(\boldsymbol{\theta}^0\right)^{-1}\right]^{-1}, \tag{18a}$$

$$\mathrm{m}^{ij} = \mathbf{C}\left(\boldsymbol{\theta}^i\right)^{-1}\mathbf{m}\left(\boldsymbol{\theta}^i\right) + \mathbf{C}\left(\boldsymbol{\theta}^j\right)^{-1}\mathbf{m}\left(\boldsymbol{\theta}^j\right) - \mathbf{C}\left(\boldsymbol{\theta}^0\right)^{-1}\mathbf{m}\left(\boldsymbol{\theta}^0\right), \tag{18b}$$

$$\delta^{ij} = \mathbf{m}\left(\boldsymbol{\theta}^i\right)^\top\mathbf{C}\left(\boldsymbol{\theta}^i\right)^{-1}\mathbf{m}\left(\boldsymbol{\theta}^i\right) + \mathbf{m}\left(\boldsymbol{\theta}^j\right)^\top\mathbf{C}\left(\boldsymbol{\theta}^j\right)^{-1}\mathbf{m}\left(\boldsymbol{\theta}^j\right) - \mathbf{m}\left(\boldsymbol{\theta}^0\right)^\top\mathbf{C}\left(\boldsymbol{\theta}^0\right)^{-1}\mathbf{m}\left(\boldsymbol{\theta}^0\right), \tag{18c}$$

then we can obtain the different components required to compute the CRB (7) (see Appendix B for detailed derivation of $\mathbf{MS}_{\mathbf{z}|\boldsymbol{\theta}}\left(\boldsymbol{\theta}^0\right)$ and $\mathbf{h}\left(\boldsymbol{\theta}^0, \boldsymbol{\theta}^j\right)$) as

$$\left[\mathbf{MS}_{\mathbf{z}|\boldsymbol{\theta}}\left(\boldsymbol{\theta}^0\right)\right]_{i,j} = \sqrt{\frac{\left|\mathbf{C}^{ij}\right|\left|\mathbf{C}(\boldsymbol{\theta}^0)\right|}{\left|\mathbf{C}(\boldsymbol{\theta}^i)\right|\left|\mathbf{C}(\boldsymbol{\theta}^j)\right|}}\, e^{\frac{1}{2}\left[\left(\mathbf{m}^{ij}\right)^\top\mathbf{C}^{ij}\mathbf{m}^{ij} - \delta^{ij}\right]} - 1, \tag{18d}$$

$$\left[\mathbf{h}\left(\boldsymbol{\theta}^0, \boldsymbol{\theta}^j\right)\right]_k = \begin{pmatrix} \frac{1}{2}tr\left(\frac{\partial \mathbf{C}(\boldsymbol{\theta}^0)^{-1}}{\partial \omega_k}\left(\mathbf{C}\left(\boldsymbol{\theta}^0\right) - \mathbf{C}\left(\boldsymbol{\theta}^j\right)\right) - \frac{\partial \mathbf{C}(\boldsymbol{\theta}^0)^{-1}}{\partial \omega_k}\right. \\ \left.\times\left(\mathbf{m}\left(\boldsymbol{\theta}^j\right) - \mathbf{m}\left(\boldsymbol{\theta}^0\right)\right)\left(\mathbf{m}\left(\boldsymbol{\theta}^j\right) - \mathbf{m}\left(\boldsymbol{\theta}^0\right)\right)^\top\right) \\ + \frac{\partial \mathbf{m}(\boldsymbol{\theta}^0)}{\partial \omega_k}^\top \mathbf{C}\left(\boldsymbol{\theta}^0\right)^{-1}\left(\mathbf{m}\left(\boldsymbol{\theta}^j\right) - \mathbf{m}\left(\boldsymbol{\theta}^0\right)\right)\end{pmatrix} \tag{18e}$$

$$\left[\mathbf{F}_{\boldsymbol{\omega}|\boldsymbol{\theta}}\left(\boldsymbol{\theta}^0\right)\right]_{k,l} = \frac{\partial \mathbf{m}\left(\boldsymbol{\theta}^0\right)}{\partial \omega_k}^\top \mathbf{C}^{-1}\left(\boldsymbol{\theta}^0\right)\frac{\partial \mathbf{m}\left(\boldsymbol{\theta}^0\right)}{\partial \omega_l} + \frac{1}{2}tr\left(\mathbf{C}^{-1}\left(\boldsymbol{\theta}^0\right)\frac{\partial \mathbf{C}\left(\boldsymbol{\theta}^0\right)}{\partial \omega_k}\mathbf{C}^{-1}\left(\boldsymbol{\theta}^0\right)\frac{\partial \mathbf{C}\left(\boldsymbol{\theta}^0\right)}{\partial \omega_l}\right), \tag{18f}$$

where (18f) is the Slepian-Bangs formula [31, p.47].

In the following, for sake of legibility, $\boldsymbol{\theta}$ denotes either the vector of unknown parameters or a selected vector value $\left(\boldsymbol{\theta}^0\right)^\top = [\left(\boldsymbol{\omega}^0\right)^\top, \left(\mathbf{z}^0\right)^\top]$.

3.3. Asymptotically Efficient Estimators for the Gaussian Linear Conditional Observation Model ($\mathbf{g}(\boldsymbol{\theta}) = \boldsymbol{\theta}$)

A case of particular interest is the Gaussian linear conditional signal model, also known as the mixed-integer model [1, Ch. 23],

$$\mathbf{y} = \mathbb{B}\boldsymbol{\omega} + \mathbb{A}\mathbf{z} + \mathbf{n}, \quad \mathbf{n} \sim \mathcal{N}_M\left(\mathbf{0}, \mathbf{C}_{\mathbf{n}}\right), \quad \boldsymbol{\theta}^\top = [\boldsymbol{\omega}^\top, \mathbf{z}^\top],\ \boldsymbol{\omega} \in \mathbb{R}^{K_\omega},\ \mathbf{z} \in \mathbb{Z}^{K_z}, \tag{19}$$

where the noise covariance matrix $\mathbf{C}_{\mathbf{n}}$ is known and the parameter vector of interest is $\mathbf{g}(\boldsymbol{\theta}) = \boldsymbol{\theta}$. For instance, in GNSS RTK precise positioning, the M-vector $\mathbf{y}$ contains the pseudorange and carrier-phase observables, the K_z-vector $\mathbf{z}$ the integer ambiguities, and the real-valued K_ω-vector $\boldsymbol{\omega}$ the remaining unknown parameters, such as, for example, position coordinates, atmospheric delay parameters (troposphere, ionosphere) and clock parameters. The theory that underpins the resolution of (19) in the maximum likelihood sense is the theory of integer inference [2][3][1, Ch. 23]. The search for the MLE of a selected value $\boldsymbol{\theta}$ for the mixed-integer model (19) can be cast as a minimization problem over mixed integer-real parameters,

$$\widehat{\boldsymbol{\theta}} = \arg\min_{\boldsymbol{\omega}\in\mathbb{R}^{K_\omega},\ \mathbf{z}\in\mathbb{Z}^{K_z}} \left\|\mathbf{y} - \mathbb{D}\boldsymbol{\theta}\right\|^2_{\mathbf{C}_{\mathbf{n}}},\ \mathbb{D} = \begin{bmatrix}\mathbb{B} & \mathbb{A}\end{bmatrix}. \tag{20}$$

A closed-form solution to (20) is not known, due to the integer nature of $\mathbf{z}$. Instead, a three-step decomposition of the problem is typically considered [32], and the resulting minimization problems are sequentially resolved as [33]

$$\min_{\boldsymbol{\omega}\in\mathbb{R}^{K_\omega},\ \mathbf{z}\in\mathbb{Z}^{K_z}} \left\|\mathbf{y} - \mathbb{D}\boldsymbol{\theta}\right\|^2_{\mathbf{C}_{\mathbf{n}}} = \left\|\mathbf{y} - \mathbb{D}\begin{pmatrix}\bar{\boldsymbol{\omega}}\\ \bar{\mathbf{z}}\end{pmatrix}\right\|^2_{\mathbf{C}_{\mathbf{n}}} \tag{21a}$$

$$+ \min_{\mathbf{z}\in\mathbb{Z}^{K_z}} \left\|\bar{\mathbf{z}} - \mathbf{z}\right\|^2_{\mathbf{C}_{\bar{\boldsymbol{\omega}}}} \tag{21b}$$

$$+ \min_{\boldsymbol{\omega}\in\mathbb{R}^{K_\omega}} \left\|\widehat{\boldsymbol{\omega}}(\mathbf{z}) - \boldsymbol{\omega}\right\|^2_{\mathbf{C}_{\widehat{\boldsymbol{\omega}}(\mathbf{z})}}, \tag{21c}$$

where $\widehat{\omega}(\mathbf{z}) = \bar{\omega} - \mathbf{C}_{\bar{\omega},\bar{\mathbf{z}}}\mathbf{C}_{\bar{\mathbf{z}}}^{-1}(\bar{\mathbf{z}} - \mathbf{z})$. The first term (21a) corresponds to the MLE solution where $\mathbf{z}$ is treated as a real valued vector (instead of an integer valued vector). The output of this estimate $\bar{\boldsymbol{\theta}}^\top = [\bar{\boldsymbol{\omega}}^\top, \bar{\mathbf{z}}^\top]$ is referred to as *float solution* and its associated covariance matrix is

$$\mathbf{C}_{\bar{\boldsymbol{\theta}}} = \begin{bmatrix} \mathbf{C}_{\bar{\boldsymbol{\omega}}} & \mathbf{C}_{\bar{\boldsymbol{\omega}},\bar{\mathbf{z}}} \\ \mathbf{C}_{\bar{\mathbf{z}},\bar{\boldsymbol{\omega}}} & \mathbf{C}_{\bar{\mathbf{z}}} \end{bmatrix} = \left(\mathbb{D}^\top \mathbf{C}_{\mathbf{n}}^{-1} \mathbb{D}\right)^{-1},$$

which, by exploiting the four-blocks matrix inversion expression [28], leads to

$$\mathbf{C}_{\widehat{\boldsymbol{\omega}}(\mathbf{z})} = \mathbf{C}_{\bar{\boldsymbol{\omega}}} - \mathbf{C}_{\bar{\boldsymbol{\omega}},\bar{\mathbf{z}}}\mathbf{C}_{\bar{\mathbf{z}}}^{-1}\mathbf{C}_{\bar{\mathbf{z}},\bar{\boldsymbol{\omega}}} = \left(\mathbb{B}^\top \mathbf{C}_{\mathbf{n}}^{-1} \mathbb{B}\right)^{-1}.$$

The second term (21b) in the decomposition corresponds to the integer-least-square (ILS), for which an integer solution $\widehat{\mathbf{z}}$ is found. Finally, the third term (21c) is the *fixed solution*, consisting on enhancing the estimates $\bar{\boldsymbol{\omega}}$ upon the estimated integer vector $\widehat{\mathbf{z}}$

$$\widehat{\boldsymbol{\omega}} = \widehat{\boldsymbol{\omega}}(\widehat{\mathbf{z}}) = \bar{\boldsymbol{\omega}} - \mathbf{C}_{\bar{\boldsymbol{\omega}},\bar{\mathbf{z}}}\mathbf{C}_{\bar{\mathbf{z}}}^{-1}(\bar{\mathbf{z}} - \widehat{\mathbf{z}}). \tag{22}$$

The improvement in $\widehat{\boldsymbol{\omega}}$ accuracy is due to constraining the float solution $\bar{\mathbf{z}}$ to a more restrictive class of estimators. Three different classes of estimators have been developed for mixed integer models [34], and for each one the optimal estimators have been identified: i) the class of integer (I) estimators [35]; ii) the class of integer-aperture (IA) estimators [36]; and iii) the class of integer-equivariant (IE) estimators [37]. The first class is the most restrictive class. This is due to the fact that the outcomes of any estimator within this class are required to be integers. Well-known examples of estimators from this class are integer rounding ($\widehat{\mathbf{z}}_R$), integer bootstrapping ($\widehat{\mathbf{z}}_B$) and the optimal solution so-called integer least-squares ($\widehat{\mathbf{z}}_{LS}$) which is the MLE. The most relaxed class is the class of IE-estimators. These estimators are real-valued, and they only obey the integer remove-restore principle. An important estimator in this class is the best IE-estimator ($\widehat{\mathbf{z}}_{BIE}$) since it has the smallest variance, even smaller than the variance of the float solution. The class of IA-estimators is a subset of the IE-estimators but it encompasses the class of I-estimators. The IA-estimators are of a hybrid nature in the sense that their outcomes are either integers or real. It is also worth noting that distributional results are readily available [25][38]. Interestingly enough, integer rounding, integer bootstrapping, and integer least-squares estimators ($\widehat{\mathbf{z}}_R$, $\widehat{\mathbf{z}}_B$, $\widehat{\mathbf{z}}_{LS}$) are uniformly unbiased [39] under Gaussian additive noise (19), leading to an uniformly unbiased estimator $\widehat{\boldsymbol{\omega}}$ (22), since then $\mathbb{E}[\widehat{\boldsymbol{\omega}}] = \mathbb{E}[\bar{\boldsymbol{\omega}}] = \boldsymbol{\omega}$. Thus the proposed $\mathbf{CRB}_{\boldsymbol{\theta}|\boldsymbol{\theta}}(\boldsymbol{\theta})$ (7) is a relevant LB for the Gaussian linear conditional signal model (19) and

$$\mathbf{C}_{\widehat{\boldsymbol{\omega}}} = \mathbf{C}_{\widehat{\boldsymbol{\omega}}(\widehat{\mathbf{z}})} \geq \mathbf{CRB}_{\boldsymbol{\omega}|\boldsymbol{\theta}}(\boldsymbol{\theta}),\ \widehat{\mathbf{z}} \in \{\widehat{\mathbf{z}}_R, \widehat{\mathbf{z}}_B, \widehat{\mathbf{z}}_{LS}\}.$$

Firstly, as [1, (23.54)] $P(\widehat{\mathbf{z}}_{LS} = \mathbf{z}) \geq P(\widehat{\mathbf{z}}_B = \mathbf{z}) \geq P(\widehat{\mathbf{z}}_R = \mathbf{z})$, and [1, (23.23)] $\lim_{tr(\mathbf{C_n})\to 0} P(\widehat{\mathbf{z}}_R = \mathbf{z}) = 1$, then

$$\lim_{tr(\mathbf{C_n})\to 0} \mathbf{C}_{\widehat{\mathbf{z}}_{LS}} = \lim_{tr(\mathbf{C_n})\to 0} \mathbf{C}_{\widehat{\mathbf{z}}_B} = \lim_{tr(\mathbf{C_n})\to 0} \mathbf{C}_{\widehat{\mathbf{z}}_R} = \mathbf{0}.$$

Thus, for any $\widehat{\mathbf{z}} \in \{\widehat{\mathbf{z}}_R, \widehat{\mathbf{z}}_B, \widehat{\mathbf{z}}_{LS}\}$, since [25, (29)] $\mathbf{C}_{\widehat{\boldsymbol{\omega}}(\widehat{\mathbf{z}})} = \mathbf{C}_{\widehat{\boldsymbol{\omega}}(\mathbf{z})} + \mathbf{C}_{\bar{\boldsymbol{\omega}},\bar{\mathbf{z}}}\mathbf{C}_{\bar{\mathbf{z}}}^{-1}\mathbf{C}_{\widehat{\mathbf{z}}}\mathbf{C}_{\bar{\mathbf{z}}}^{-1}\mathbf{C}_{\bar{\mathbf{z}},\bar{\boldsymbol{\omega}}}$,

$$\lim_{tr(\mathbf{C_n})\to 0} \mathbf{C}_{\widehat{\boldsymbol{\omega}}(\widehat{\mathbf{z}})} = \left(\mathbb{B}^\top \mathbf{C}_{\mathbf{n}}^{-1} \mathbb{B}\right)^{-1}.$$

Secondly, since it is well known that adding unknown parameters leads to an equal or higher CRB, then (25a)

$$\mathbf{C}_{\widehat{\boldsymbol{\omega}}(\widehat{\mathbf{z}})} \geq \mathbf{CRB}_{\boldsymbol{\omega}|\boldsymbol{\theta}}(\boldsymbol{\theta}) \geq \mathbf{F}_{\boldsymbol{\omega}|\boldsymbol{\theta}}^{-1}(\boldsymbol{\theta}) = \left(\mathbb{B}^\top \mathbf{C}_{\mathbf{n}}^{-1} \mathbb{B}\right)^{-1}.$$

Therefore, for any $\widehat{\mathbf{z}} \in \{\widehat{\mathbf{z}}_R, \widehat{\mathbf{z}}_B, \widehat{\mathbf{z}}_{LS}\}$,

$$\lim_{tr(\mathbf{C_n})\to 0} \mathbf{C}_{\widehat{\boldsymbol{\omega}}(\widehat{\mathbf{z}})} = \lim_{tr(\mathbf{C_n})\to 0} \mathbf{CRB}_{\boldsymbol{\omega}|\boldsymbol{\theta}}(\boldsymbol{\theta}) = \left(\mathbb{B}^\top \mathbf{C}_{\mathbf{n}}^{-1} \mathbb{B}\right)^{-1},$$

which proves that $\widehat{\mathbf{z}}_R$, $\widehat{\mathbf{z}}_B$ and $\widehat{\mathbf{z}}_{LS}$ are asymptotically efficient estimators. Last, since $\widehat{\mathbf{z}}_{BIE}$ is also uniformly unbiased with a MSE less than or at the most equal to the MSE of $\widehat{\mathbf{z}}_{LS}$ [37, (24)], it is an asymptotically efficient estimator as well.

3.4. Generalizations and Outlooks

The proposed CRB for mixed parameter vectors (7) has been derived in the context of "standard" deterministic estimation problems for which a closed-form expression of $p(\mathbf{y};\boldsymbol{\theta})$ is available. In the context of "non standard" deterministic estimation problems (see [40] and references therein), $p(\mathbf{y};\boldsymbol{\theta})$ results from the marginalization of an hybrid p.d.f. depending on both random $\left(\boldsymbol{\theta}_r \in \mathbb{R}^{P_r}\right)$ and deterministic $(\boldsymbol{\theta})$ parameters, i.e., $p(\mathbf{y};\boldsymbol{\theta}) = \int_{\mathbb{R}^{P_r}} p(\mathbf{y}, \boldsymbol{\theta}_r|\boldsymbol{\theta})\, d\boldsymbol{\theta}_r$, which is mathematically intractable and prevents from using the proposed standard CRB (7). Fortunately, any LB deriving from the MSB, as for instance (7), can be transformed into two variants, namely the so-called "modified" LB [40, Section III] and "non-standard" LB [40, Section IV] fitted to non-standard deterministic estimation. Thus, two CRB variants for mixed parameter vectors can be readily introduced in the context of "non standard" deterministic estimation. Since the proposed CRB (7) can also be regarded as a CRB dealing with restricting the set of possible values of some of the unknown parameters, namely the integer-valued parameters, it is worth noticing that (7) can also take into account continuous restriction on the set of possible values of the real-valued parameters. When the continuous restriction is described by a set of P equality constraints, $\mathbf{f}(\boldsymbol{\omega}) = \mathbf{0} \in \mathbb{R}^P$, $1 \leq P \leq K_\omega - 1$, where the matrix $\partial \mathbf{f}(\boldsymbol{\omega})/\partial \boldsymbol{\omega}^\top \in \mathbb{R}^{P \times K_\omega}$ has full row rank (P), which is equivalent to requiring that the constraints are not redundant, it leads to the so-called constrained CRB [41, 42, 43, 44]. In the case of mixed parameter vectors, it amounts to replace the LCs (13) with [29]

$$\mathbb{E}_{\mathbf{y};\boldsymbol{\theta}}\left[\left(\mathbf{U}(\boldsymbol{\omega})^\top \frac{\partial \ln p(\mathbf{y};\boldsymbol{\theta})}{\partial \boldsymbol{\omega}}\right)\left(\widehat{\mathbf{g}(\boldsymbol{\theta})}(\mathbf{y}) - \mathbf{g}(\boldsymbol{\theta})\right)^\top\right] = \left(\frac{\partial \mathbf{g}(\boldsymbol{\theta})}{\partial \boldsymbol{\omega}^\top}\mathbf{U}(\boldsymbol{\omega})\right)^\top,$$

where $\mathbf{U}(\boldsymbol{\omega}) \in \mathbb{R}^{K_\omega \times (K_\omega - P)}$ is a basis of $\ker\left\{\frac{\partial \mathbf{f}(\boldsymbol{\omega})}{\partial \boldsymbol{\omega}^\top}\right\}$, which leads to update the definition of $\mathbf{F}_{\boldsymbol{\omega}|\boldsymbol{\theta}}(\boldsymbol{\theta})$ (11a), $\mathbf{H}(\boldsymbol{\theta})$ (11b) and $\boldsymbol{\Lambda}_{\mathbf{g}}(\boldsymbol{\theta})$ (8) as follows: $\mathbf{F}_{\boldsymbol{\omega}|\boldsymbol{\theta}}(\boldsymbol{\theta}) \to \mathbf{U}(\boldsymbol{\omega})^\top \mathbf{F}_{\boldsymbol{\omega}|\boldsymbol{\theta}}(\boldsymbol{\theta}) \mathbf{U}(\boldsymbol{\omega})$, $\mathbf{H}(\boldsymbol{\theta}) \to \mathbf{U}(\boldsymbol{\omega})^\top \mathbf{H}(\boldsymbol{\theta})$, and $\boldsymbol{\Lambda}_{\mathbf{g}}(\boldsymbol{\theta}) = \left[\frac{\partial \mathbf{g}(\boldsymbol{\theta})}{\partial \boldsymbol{\omega}^\top}\mathbf{U}(\boldsymbol{\omega}) \ \mathbf{g}(\boldsymbol{\theta}^1) - \mathbf{g}(\boldsymbol{\theta}^0) \ \ldots \ \mathbf{g}(\boldsymbol{\theta}^{2K_z}) - \mathbf{g}(\boldsymbol{\theta}^0)\right]$.

4. Example of Gaussian Linear Regression Problem: GNSS RTK Precise Positioning

In this section, we exemplify the aforementioned results with a particular example of Gaussian linear conditional signal model, aka the linear regression problem with mixed real and integer parameters, that is the GNSS RTK precise positioning problem.

4.1. Signal Model for GNSS RTK Precise Positioning

RTK is a GNSS-based relative positioning method, where the position of a target is determined with respect to a base station of known coordinates [45]. RTK exploits the use of code and carrier-phase pseudorange observations. Carrier-phase observations are characterized by a noise (typically) two orders of magnitude lower than code pseudorange measurement, but they are ambiguous by an unknown number of integer ambiguities. Thus, the achievement of high precision positioning requires the estimation of the dynamical parameters of the target along with the unknown integer ambiguities within a process referred to as Integer Ambiguity Resolution (IAR) [33]. We assume that $M+1$ satellites are tracked simultaneously by the base and target GNSS receivers. First, single-differencing the observations, i.e., subtracting the observations at the target from the base stations, eliminates the atmospheric and satellite-related delays. Then, the process of double-differencing, i.e., subtracting the single-difference observations with respect to a reference satellite, removes the effects of receivers clock offsets. The resulting GNSS RTK model can be modeled as $\mathbf{y} \sim \mathcal{N}_M(\mathbf{m}(\boldsymbol{\theta}), \mathbf{C}_\mathbf{n})$ with

$$\mathbf{m}(\boldsymbol{\theta}) = \mathbb{D}\begin{bmatrix}\boldsymbol{\omega}\\ \mathbf{z}\end{bmatrix},\ \mathbb{D} = [\mathbb{B}\ \mathbb{A}],$$
$$\mathbb{B} = \begin{bmatrix}\mathbf{B}\\ \mathbf{B}\end{bmatrix},\ \mathbb{A} = \begin{bmatrix}\mathbf{A}\\ \mathbf{0}\end{bmatrix},\ \mathbf{C}_\mathbf{n} = \begin{bmatrix}\mathbf{C}_\phi & \mathbf{0}\\ \mathbf{0} & \mathbf{C}_\rho\end{bmatrix}, \tag{23}$$

where $\mathbf{y}^\top = \left[\boldsymbol{\phi}^\top, \boldsymbol{\rho}^\top\right]$ is the observation vector, composed by the double-difference carrier-phase and code observations, denoted as $\boldsymbol{\phi}, \boldsymbol{\rho} \in \mathbb{R}^M$ respectively, whose corresponding covariance matrices are $\mathbf{C}_\phi$ and $\mathbf{C}_\rho$. $\boldsymbol{\omega} \in \mathbb{R}^3$ is the target receiver to base station baseline vector; and $\mathbf{z} \in \mathbb{Z}^M$ is the vector of unknown integer ambiguities. The matrix $\mathbf{B}$ is the so-called geometry matrix which is composed of the unit line-of-sight vectors pointing from the receiver to each satellite. $\mathbf{A}$ is the diagonal matrix with the wavelength of the carrier-phase measurements [32, 46]. The covariance matrices $\mathbf{C}_\phi$ and $\mathbf{C}_\rho$ are defined as $\mathbf{C}_\phi = 2\sigma_\phi^2 \mathbf{T}\mathbf{W}^{-1}\mathbf{T}^\top$ and $\mathbf{C}_\rho = 2\sigma_\rho^2 \mathbf{T}\mathbf{W}^{-1}\mathbf{T}^\top$, where σ_ϕ and σ_ρ are the zenith-referenced undifferenced phase and code standard deviations [46], $\mathbf{T} = [\mathbf{I}_M \ \ -\mathbf{1}_M]$ is the double-differencing matrix, $\mathbf{W} = \mathrm{diag}(w_1, \ldots, w_{M+1})$ is a diagonal values and w_i is the satellite elevation-dependent weight. As it is formulated, the RTK problem can be seen to fit the linear regression problem discussed earlier in Section 4. In practice, the solution for the mixed real and integer model (20) is generally solved via the mixed real-integer regression (21a-21c). Thus, the RTK positioning model constitutes a practical example of Gaussian linear regression with mixed real- and integer-valued parameters.

4.2. CRB for GNSS RTK Precise Positioning

As aforementioned, in the RTK problem, the Gaussian linear observation model reads

$$\mathbf{y} = \begin{bmatrix}\boldsymbol{\phi}\\ \boldsymbol{\rho}\end{bmatrix} = \mathbb{D}\boldsymbol{\theta} + \mathbf{n},\ \mathbf{n} \sim \mathcal{N}_M(\mathbf{0}, \mathbf{C}_\mathbf{n}),$$
$$\mathbf{B} = [\mathbf{b}_1 \ldots \mathbf{b}_{K_\omega}], \mathbf{A} = [\mathbf{a}_1 \ldots \mathbf{a}_{K_z}]. \tag{24}$$

From the results presented in Section 3.1, we recall that the CRB, referred to as $\mathrm{CRB}_{Real/Integer}$ in the following, is given by

$$\mathbf{CRB}_{\boldsymbol{\theta}|\boldsymbol{\theta}}(\boldsymbol{\theta}) = \boldsymbol{\Lambda}_{\boldsymbol{\theta}}(\boldsymbol{\theta}) \overline{\mathbf{F}}(\boldsymbol{\theta})^{-1} \boldsymbol{\Lambda}_{\boldsymbol{\theta}}^\top(\boldsymbol{\theta}),$$
$$\overline{\mathbf{F}}(\boldsymbol{\theta}) = \begin{bmatrix}\mathbf{F}_{\boldsymbol{\omega}|\boldsymbol{\theta}}(\boldsymbol{\theta}) & \mathbf{H}(\boldsymbol{\theta})\\ \mathbf{H}(\boldsymbol{\theta})^\top & \mathbf{MS}_{\mathbf{z}|\boldsymbol{\theta}}(\boldsymbol{\theta})\end{bmatrix},$$

where $\boldsymbol{\Lambda}_{\boldsymbol{\theta}}(\boldsymbol{\theta})$ is given by (16), and we have to compute $\overline{\mathbf{F}}(\boldsymbol{\theta})$ using the general Gaussian model equations given in Section 3.2. Since $\mathbf{C}_\mathbf{n}$ does not depend on $\boldsymbol{\theta}$, (18a-18c) become

$$\mathbf{C}^{ij} = \mathbf{C}_\mathbf{n},\ \mathbf{m}^{ij} = \mathbf{C}_\mathbf{n}^{-1}\left(\mathbf{m}(\boldsymbol{\theta}^i) + \mathbf{m}(\boldsymbol{\theta}^j) - \mathbf{m}(\boldsymbol{\theta})\right),$$
$$\delta^{ij} = \mathbf{m}(\boldsymbol{\theta}^i)^\top \mathbf{C}_\mathbf{n}^{-1}\mathbf{m}(\boldsymbol{\theta}^i) + \mathbf{m}(\boldsymbol{\theta}^j)^\top \mathbf{C}_\mathbf{n}^{-1}\mathbf{m}(\boldsymbol{\theta}^j) - \mathbf{m}(\boldsymbol{\theta})^\top \mathbf{C}_\mathbf{n}^{-1}\mathbf{m}(\boldsymbol{\theta}),$$
$$[\mathbf{MS}]_{i,j} = e^{\left(\mathbf{m}(\boldsymbol{\theta}) - \mathbf{m}(\boldsymbol{\theta}^i) - \mathbf{m}(\boldsymbol{\theta}^j)\right)^\top \mathbf{C}_\mathbf{n}^{-1}\mathbf{m}(\boldsymbol{\theta}) + \mathbf{m}(\boldsymbol{\theta}^i)^\top \mathbf{C}_\mathbf{n}^{-1}\mathbf{m}(\boldsymbol{\theta}^j)}.$$

These equations allow computation of the elements in the CRB formula. Particularly, computing $\mathbf{F}_{\boldsymbol{\omega}|\boldsymbol{\theta}}(\boldsymbol{\theta})$ for $1 \leq k, k' \leq K_\omega$ becomes

$$\left[\mathbf{F}_{\boldsymbol{\omega}|\boldsymbol{\theta}}(\boldsymbol{\theta})\right]_{k,k'} = \frac{\partial \mathbf{m}(\boldsymbol{\theta})}{\partial \omega_k}^\top \mathbf{C}_\mathbf{n}^{-1} \frac{\partial \mathbf{m}(\boldsymbol{\theta})}{\partial \omega_{k'}} = \begin{bmatrix}\mathbf{b}_k\\ \mathbf{b}_k\end{bmatrix}^\top \mathbf{C}_\mathbf{n}^{-1} \begin{bmatrix}\mathbf{b}_{k'}\\ \mathbf{b}_{k'}\end{bmatrix},$$

such that

$$\mathbf{F}_{\boldsymbol{\omega}|\boldsymbol{\theta}}(\boldsymbol{\theta}) = \mathbb{B}^\top \mathbf{C}_\mathbf{n}^{-1} \mathbb{B}. \tag{25a}$$

Let $\boldsymbol{\theta}^j = \boldsymbol{\theta} + (-1)^{j-1}\mathbf{i}_{K_\omega + \lfloor \frac{j+1}{2} \rfloor}$ and $\boldsymbol{\theta}^i = \boldsymbol{\theta} + (-1)^{i-1}\mathbf{i}_{K_\omega + \lfloor \frac{i+1}{2} \rfloor}$. Then,

$$\left[\mathbf{MS}_{\mathbf{z}|\boldsymbol{\theta}}(\boldsymbol{\theta})\right]_{i,j} = e^{\left(\boldsymbol{\theta} - \boldsymbol{\theta}^i - \boldsymbol{\theta}^j\right)^\top \mathbb{D}^\top \mathbf{C}_\mathbf{n}^{-1} \mathbb{D}\theta + \left(\boldsymbol{\theta}^i\right)^\top \mathbb{D}^\top \mathbf{C}_\mathbf{n}^{-1} \mathbb{D}\left(\boldsymbol{\theta}^j\right)} - 1, \tag{25b}$$

and for $1 \leq k \leq K_\omega$, we have that

$$\left[\mathbf{h}\left(\boldsymbol{\theta}, \boldsymbol{\theta}^j\right)\right]_k = \frac{\partial \mathbf{m}\left(\boldsymbol{\theta}\right)}{\partial \omega_k}^\top \mathbf{C}_\mathbf{n}^{-1}\left(\mathbf{m}\left(\boldsymbol{\theta}^j\right) - \mathbf{m}\left(\boldsymbol{\theta}\right)\right) = \begin{bmatrix} \mathbf{b}_k \\ \mathbf{b}_k \end{bmatrix}^\top \mathbf{C}_\mathbf{n}^{-1}\mathbb{D}\left(\boldsymbol{\theta}^j - \boldsymbol{\theta}\right),$$

which leads to

$$\mathbf{H}\left(\boldsymbol{\theta}\right) = \mathbb{B}^\top \mathbf{C}_\mathbf{n}^{-1}\mathbb{D}\begin{bmatrix} \mathbf{i}_{K_\omega+1} & -\mathbf{i}_{K_\omega+1} & \dots & \mathbf{i}_K & -\mathbf{i}_K \end{bmatrix}. \tag{25c}$$

It is worth remembering that relaxing the condition on the integer-valued part of the parameters' vector, and assuming that both parameters are real-valued, $\boldsymbol{\omega} \in \mathbb{R}^{K_\omega}$, $\mathbf{z} \in \mathbb{R}^{K_z}$, then the standard CRB is given by the inverse of the following FIM,

$$\mathbf{F}_{\boldsymbol{\theta}|\boldsymbol{\theta}}\left(\boldsymbol{\theta}\right) = \mathbb{D}^\top \mathbf{C}_\mathbf{n}^{-1}\mathbb{D}, \tag{26}$$

and is referred to as CRB_{Real} in the following.

4.3. Illustrative Example

To illustrate the validity of the proposed LB, a realistic GNSS RTK experiment was simulated. Particularly, the receiver-satellite geometry considered is illustrated in Fig. 1 ($M+1 = 13$ satellites), under a wide range of precision levels for the undifferenced code observations –preserving the noise of carrier-phase ($\boldsymbol{\phi}$) measurements two orders of magnitude lower than code ($\boldsymbol{\rho}$) observable –. To evaluate the LS performance, the root (total) MSE (RMSE), obtained from 10^4 Monte Carlo runs, for both 3D position and the 12 (M) integer ambiguities was considered as a measure of performance.

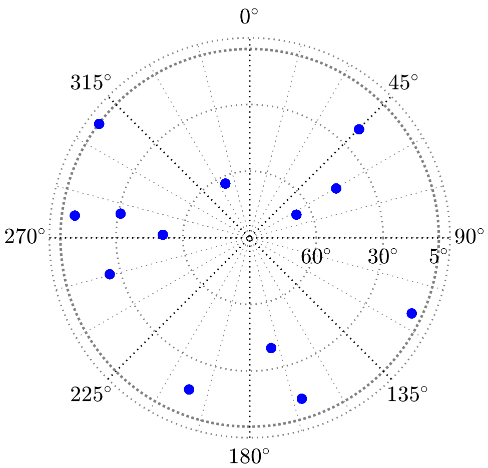

Figure 1: Skyplot of the simulated GNSS receiver-satellite geometry.

Fig. 2 (left) shows the 3D position ($\boldsymbol{\omega}$) RMSE over the standard deviation of code observations σ_ρ, as well as the square-root of the corresponding (total) CRBs, with a zoom of the low noise region given in Fig. 2 (right). First, notice that the standard LS (equivalent to the MLE) RMSE, as expected, coincides with the CRB_{Real} for the range of tested σ_ρ values, which gives the ultimate achievable performance with both code and phase observables if no integer constraint is imposed for ambiguities $\mathbf{z}$. Secondly, the ILS performance, considering that the float position estimate is always corrected by the output ambiguities of the IAR, clearly depends on the noise level. Three regions of performance can identified: *i*) *large noise* regime: the ILS coincides with both standard LS and CRB_{Real}, which is clear from the ILS success rate shown in Fig. 3, where we can see that for $\sigma_\rho > 5$ [m] a correct integer solution is never found, then, on average, is as if no integer constraint was imposed; *ii*) *low noise* regime: the IAR obtains the correct ambiguity solution with high probability, then the ILS coincides with the so-called *Correct ILS* (which only considers the successful outputs of the IAR) and the $\text{CRB}_{Real|Integer}$, which shows that the ILS is asymptotically efficient; and *iii*) *threshold region*: below the so-called threshold point (in this case, $\sigma_\rho > 0.1$ [m]), the ILS RMSE departs from the $\text{CRB}_{Real|Integer}$ and rises towards the CRB_{Real}, with even a small region where the RMSE overpasses the performance of a standard LS (in this case, for $1 < \sigma_\rho < 10$ [m]). This region describes the behaviour of the ILS, which abandons its asymptotic efficiency and ambiguous errors occur due to the (partially) wrong estimation of the integer ambiguities. The threshold point varies with the satellite geometry, number of observations (*i.e.,* number of frequencies tracked) and observation noises. Therefore, the precise prediction for the transition point remains an open challenge. Finally, if we considered only the successfully fixed ambiguities, the *Correct ILS* would coincide with the $\text{CRB}_{Real|Integer}$. However, the correct solution to the ILS problem cannot be guaranteed outside the asymptotic region.

Regarding the $\text{CRB}_{Real|Integer}$ and CRB_{Real} comparison, it is clear that considering the integer nature of a part of the vector to be estimated has a strong impact on the achievable performance, and therefore, highlights the interest of the estimating the so-called integer ambiguities. As a byproduct, this highlights the importance of the LB proposed in this contribution. Obviously, restricting the set of possible values (integer instead of real) leads to a LB such that $\text{CRB}_{Real|Integer} \leq \text{CRB}_{Real}$. For this bound there exists also a noise *threshold region* from where the real/integer parameters bound meets the real parameters bound. This implies that in such high noise region the integer constraint does not improve the estimates of the real parameters.

Fig. 4 (top) shows the ambiguity ($\mathbf{z}$) RMSE as a function of the standard deviation of the code observations σ_ρ (recall that σ_ϕ is always set two orders of magnitude lower than σ_ρ in these simulations), as well as the square-root of the corresponding CRBs, with a zoom of the low noise region given in Fig. 4 (bottom). Again, we can identify the same behaviour as for the position estimate: *i*) the standard LS ambiguity estimation coincides with the CRB_{Real}; *ii*) in the high noise region, the IAR output (i.e., all ambiguities) coincides with the real ambiguity case; and

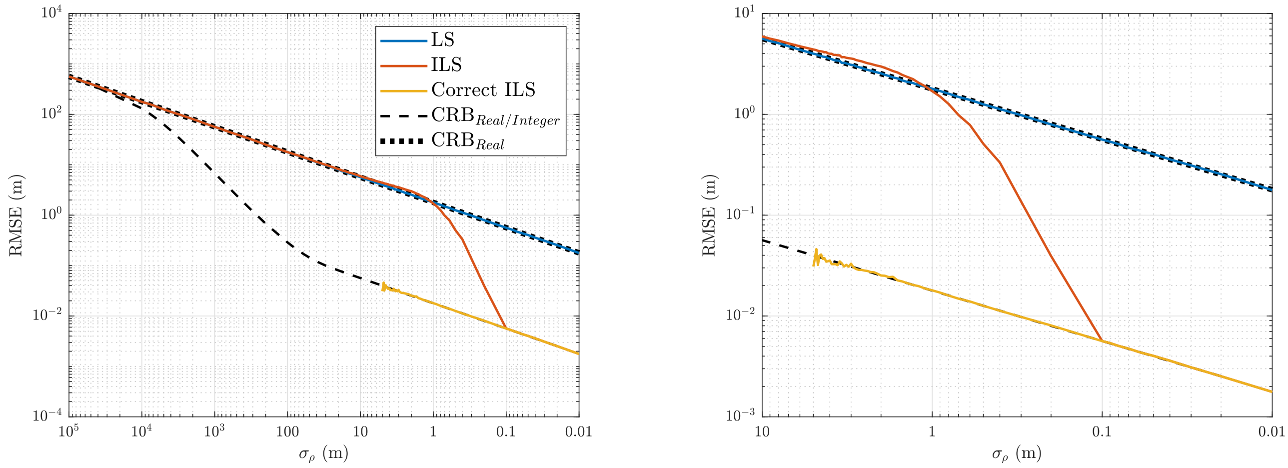

Figure 2: Positioning RMSE and square-root of CRBs as a function of the standard deviation of observation noise σ_ρ.

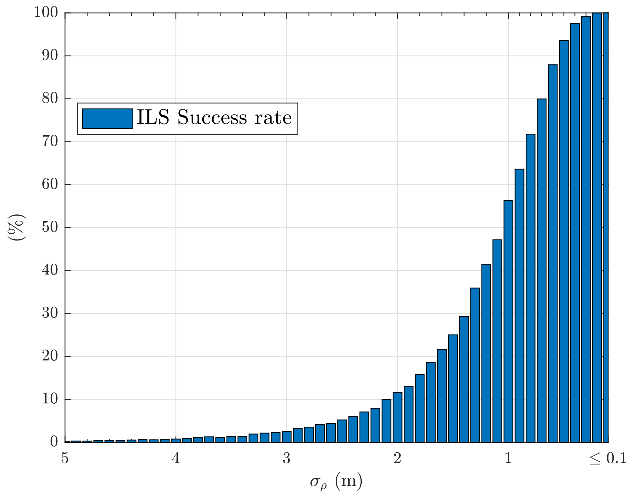

Figure 3: Success rate for the integer parameter estimation of the RTK positioning problem.

iii) when the success rate increases (i.e., for $1 > \sigma_\rho > 0.2$), the ILS ambiguity RMSE tends to decrease until the point where all the ambiguities are correctly fixed ($\sigma_\rho \leq 0.2$) and then the RMSE coincides with the $\text{CRB}_{Real|Integer} = 0$, being in the asymptotic efficiency region. The *Correct ILS* is not shown because both RMSE and $\text{CRB}_{Real|Integer}$ are equal to 0. Together with the previous results for the position estimate, this shows the validity and interest of the mixed real/integer bound, and the consistency of the results related to the ambiguity fixing capabilities (i.e., success rate).

5. Conclusions

The main object of this contribution was the derivation of LBs on the estimation of mixed real- and integer-valued parameter vectors. A closed-form Cramér-Rao bound (CRB) for this problem was provided, which leverages the general CRB expression as the limiting form of the McAulay-Seidman bound. The general CRB expression for mixed parameter vectors was particularized for the Gaussian observation problem. To show the validity of the bound derived in the article, results for a representative carrier phase-based precise positioning example were provided. It was shown that the CRB expression is able to predict the RMSE performance of the MLE, and that an asymptotically efficient estimator for mixed parameter vectors exists in linear regression model with known noise covariance matrix.

Appendix A. Proof of (5a)-(5c)

Let $\boldsymbol{\varepsilon}_{\mathbf{g}}\left(\mathbf{y};\boldsymbol{\theta}^0\right) = \widehat{\mathbf{g}\left(\boldsymbol{\theta}^0\right)}\left(\mathbf{y}\right) - \mathbf{g}\left(\boldsymbol{\theta}^0\right)$. From Lemma 3 in [29], the set of linear constraints

$$\mathbb{E}_{\mathbf{y};\boldsymbol{\theta}^0}\left[\boldsymbol{\upsilon}_{\boldsymbol{\theta}^0}\left(\mathbf{y};\{\boldsymbol{\theta}\}^{1+K}\right)\boldsymbol{\varepsilon}_{\mathbf{g}}^\top\left(\mathbf{y};\boldsymbol{\theta}^0\right)\right] = \begin{bmatrix} \mathbf{0}^\top \\ \left(\mathbf{g}\left(\boldsymbol{\theta}^0 + \mathbf{i}_1 d\theta_1\right) - \mathbf{g}\left(\boldsymbol{\theta}^0\right)\right)^\top \\ \vdots \\ \left(\mathbf{g}\left(\boldsymbol{\theta}^0 + \mathbf{i}_K d\theta_K\right) - \mathbf{g}\left(\boldsymbol{\theta}^0\right)\right)^\top \end{bmatrix} = \mathbf{V}, \quad \text{(A.1)}$$

are equivalent to

$$\mathbf{T}^\top\mathbb{E}_{\mathbf{y};\boldsymbol{\theta}^0}\left[\boldsymbol{\upsilon}_{\boldsymbol{\theta}^0}\left(\mathbf{y};\{\boldsymbol{\theta}\}^{1+K}\right)\boldsymbol{\varepsilon}_{\mathbf{g}}^\top\left(\mathbf{y};\boldsymbol{\theta}^0\right)\right] = \mathbf{T}^\top\mathbf{V},$$

where (weighted subtraction of the first constraint)

$$\mathbf{T}^\top = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ -1/d\theta_1 & 1/d\theta_1 & 0 & \dots & 0 \\ -1/d\theta_2 & 0 & 1/d\theta_2 & 0 & \vdots \\ \vdots & \vdots & 0 & \ddots & 0 \\ -1/d\theta_K & 0 & \dots & 0 & 1/d\theta_K \end{bmatrix},$$

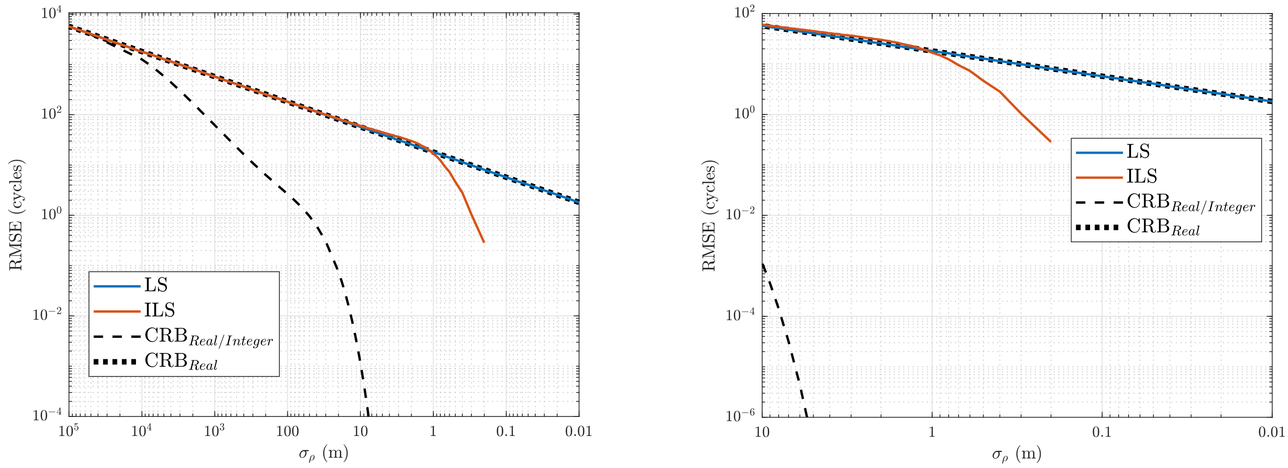

Figure 4: Ambiguity RMSE and square-root of CRBs as a function of the standard deviation of observation noise σ_ρ.

that is

$$\mathbb{E}_{\mathbf{y};\boldsymbol{\theta}^0}\left[\begin{pmatrix} 1 \\ \frac{p\left(\mathbf{y};\boldsymbol{\theta}^0+\mathbf{i}_1 d\theta_1\right)-p\left(\mathbf{y};\boldsymbol{\theta}^0\right)}{d\theta_1 p\left(\mathbf{y};\boldsymbol{\theta}^0\right)} \\ \vdots \\ \frac{p\left(\mathbf{y};\boldsymbol{\theta}^0+\mathbf{i}_K d\theta_K\right)-p\left(\mathbf{y};\boldsymbol{\theta}^0\right)}{d\theta_K p\left(\mathbf{y};\boldsymbol{\theta}^0\right)} \end{pmatrix} \boldsymbol{\varepsilon}_{\mathbf{g}}^\top\left(\mathbf{y};\boldsymbol{\theta}^0\right)\right] = \begin{bmatrix} \mathbf{0}^\top \\ \left(\frac{\mathbf{g}\left(\boldsymbol{\theta}^0+\mathbf{i}_1 d\theta_1\right)-\mathbf{g}\left(\boldsymbol{\theta}^0\right)}{d\theta_1}\right)^\top \\ \vdots \\ \left(\frac{\mathbf{g}\left(\boldsymbol{\theta}^0+\mathbf{i}_K d\theta_K\right)-\mathbf{g}\left(\boldsymbol{\theta}^0\right)}{d\theta_K}\right)^\top \end{bmatrix}. \tag{A.2}$$

Moreover, since

$$\mathbb{E}_{\mathbf{y};\boldsymbol{\theta}^0}\left[1 \times \begin{pmatrix} \frac{p\left(\mathbf{y};\boldsymbol{\theta}^0+\mathbf{i}_1 d\theta_1\right)-p\left(\mathbf{y};\boldsymbol{\theta}^0\right)}{d\theta_1 p\left(\mathbf{y};\boldsymbol{\theta}^0\right)} \\ \vdots \\ \frac{p\left(\mathbf{y};\boldsymbol{\theta}^0+\mathbf{i}_K d\theta_K\right)-p\left(\mathbf{y};\boldsymbol{\theta}^0\right)}{d\theta_K p\left(\mathbf{y};\boldsymbol{\theta}^0\right)} \end{pmatrix}\right] = \begin{bmatrix} \frac{1}{d\theta_1}\left(\mathbb{E}_{\mathbf{y};\boldsymbol{\theta}^0}\left[\frac{p\left(\mathbf{y};\boldsymbol{\theta}^0+\mathbf{i}_1 d\theta_1\right)}{p\left(\mathbf{y};\boldsymbol{\theta}^0\right)}\right]-1\right) \\ \vdots \\ \frac{1}{d\theta_K}\left(\mathbb{E}_{\mathbf{y};\boldsymbol{\theta}^0}\left[\frac{p\left(\mathbf{y};\boldsymbol{\theta}^0+\mathbf{i}_K d\theta_K\right)}{p\left(\mathbf{y};\boldsymbol{\theta}^0\right)}\right]-1\right) \end{bmatrix} = \mathbf{0},$$

we can apply Lemma 2 in [29] to assert that (A.1) and (A.2) are equivalent to

$$\mathbb{E}_{\mathbf{y};\boldsymbol{\theta}^0}\left[\begin{pmatrix} \frac{p\left(\mathbf{y};\boldsymbol{\theta}^0+\mathbf{i}_1 d\theta_1\right)-p\left(\mathbf{y};\boldsymbol{\theta}^0\right)}{d\theta_1 p\left(\mathbf{y};\boldsymbol{\theta}^0\right)} \\ \vdots \\ \frac{p\left(\mathbf{y};\boldsymbol{\theta}^0+\mathbf{i}_K d\theta_K\right)-p\left(\mathbf{y};\boldsymbol{\theta}^0\right)}{d\theta_K p\left(\mathbf{y};\boldsymbol{\theta}^0\right)} \end{pmatrix} \boldsymbol{\varepsilon}_{\mathbf{g}}^\top\left(\mathbf{y};\boldsymbol{\theta}^0\right)\right] = \begin{bmatrix} \left(\frac{\mathbf{g}\left(\boldsymbol{\theta}^0+\mathbf{i}_1 d\theta_1\right)-\mathbf{g}\left(\boldsymbol{\theta}^0\right)}{d\theta_1}\right)^\top \\ \vdots \\ \left(\frac{\mathbf{g}\left(\boldsymbol{\theta}^0+\mathbf{i}_K d\theta_K\right)-\mathbf{g}\left(\boldsymbol{\theta}^0\right)}{d\theta_K}\right)^\top \end{bmatrix}. \tag{A.3}$$

Q.E.D.

Appendix B. Derivation of (18a)-(18d)

Let us consider an M-dimensional Gaussian real vector $\mathbf{y}$ with mean $\mathbf{m}_{\mathbf{y}} = \mathbf{m}(\boldsymbol{\theta})$ and covariance matrix $\mathbf{C}_{\mathbf{y}} = \mathbf{C}(\boldsymbol{\theta})$: $\mathbf{y} \sim \mathcal{N}_M(\mathbf{m}(\boldsymbol{\theta}), \mathbf{C}(\boldsymbol{\theta}))$ and $p(\mathbf{y};\boldsymbol{\theta}) = p(\mathbf{y};\mathbf{m}(\boldsymbol{\theta}),\mathbf{C}(\boldsymbol{\theta}))$ as in (17). The derivation of the components $\mathbf{MS}_{\mathbf{z}|\boldsymbol{\theta}}$ and $\mathbf{H}_{\boldsymbol{\theta}}$ of the $\mathbf{CRB}_{\mathbf{g}|\boldsymbol{\theta}}\left(\boldsymbol{\theta}^0\right)$ in (7-11d) is based on the following factorization property of the Gaussian real pdf,

$$\frac{p(\mathbf{y};\boldsymbol{\theta}^i)p(\mathbf{y};\boldsymbol{\theta}^j)}{p(\mathbf{y};\boldsymbol{\theta}^0)} = \frac{p\left(\mathbf{y};\mathbf{m}(\boldsymbol{\theta}^i),\mathbf{C}\left(\boldsymbol{\theta}^i\right)\right)p\left(\mathbf{y};\mathbf{m}\left(\boldsymbol{\theta}^j\right),\mathbf{C}\left(\boldsymbol{\theta}^j\right)\right)}{p\left(\mathbf{y};\mathbf{m}\left(\boldsymbol{\theta}^0\right),\mathbf{C}\left(\boldsymbol{\theta}^0\right)\right)} = [\mathbf{MS}]_{i,j}\ p\left(\mathbf{y};\mathbf{C}^{ij}\mathbf{m}^{ij},\mathbf{C}^{ij}\right), \tag{B.1}$$

where

$$\mathbf{C}^{ij} = \left[\mathbf{C}\left(\boldsymbol{\theta}^i\right)^{-1} + \mathbf{C}\left(\boldsymbol{\theta}^j\right)^{-1} - \mathbf{C}\left(\boldsymbol{\theta}^0\right)^{-1}\right]^{-1}, \tag{B.2a}$$

$$\mathbf{m}^{ij} = \mathbf{C}\left(\boldsymbol{\theta}^i\right)^{-1}\mathbf{m}\left(\boldsymbol{\theta}^i\right) + \mathbf{C}\left(\boldsymbol{\theta}^j\right)^{-1}\mathbf{m}\left(\boldsymbol{\theta}^j\right) - \mathbf{C}\left(\boldsymbol{\theta}^0\right)^{-1}\mathbf{m}\left(\boldsymbol{\theta}^0\right), \tag{B.2b}$$

$$\delta^{ij} = \mathbf{m}\left(\boldsymbol{\theta}^i\right)^\top \mathbf{C}\left(\boldsymbol{\theta}^i\right)^{-1}\mathbf{m}\left(\boldsymbol{\theta}^i\right) + \mathbf{m}\left(\boldsymbol{\theta}^j\right)^\top \mathbf{C}\left(\boldsymbol{\theta}^j\right)^{-1}\mathbf{m}\left(\boldsymbol{\theta}^j\right) - \mathbf{m}\left(\boldsymbol{\theta}^0\right)^\top \mathbf{C}\left(\boldsymbol{\theta}^0\right)^{-1}\mathbf{m}\left(\boldsymbol{\theta}^0\right), \tag{B.2c}$$

$$[\mathbf{MS}]_{i,j} = \sqrt{\frac{\left|\mathbf{C}^{ij}\right|\left|\mathbf{C}\left(\boldsymbol{\theta}^0\right)\right|}{\left|\mathbf{C}\left(\boldsymbol{\theta}^i\right)\right|\left|\mathbf{C}\left(\boldsymbol{\theta}^j\right)\right|}}\, e^{\frac{1}{2}\left[\left(\mathbf{m}^{ij}\right)^\top \mathbf{C}^{ij}\mathbf{m}^{ij} - \delta^{ij}\right]}, \tag{B.2d}$$

which suggests a breakdown into items $\left(\left[\mathbf{MS}_{\mathbf{z}|\boldsymbol{\theta}}\left(\boldsymbol{\theta}^0\right)\right]_{i,j}, \mathbf{h}\left(\boldsymbol{\theta}^0, \boldsymbol{\theta}^j\right)\right)$ depending only on the selected value $\boldsymbol{\theta}^0$ and a couple of test points $\left\{\boldsymbol{\theta}^i, \boldsymbol{\theta}^j\right\}^{i,j\in[0,2K_Z]}$, as detailed in (11b-11d). Indeed, denoting

$$\mathbb{E}_{\mathbf{y}}^{ij}\left[g\left(\mathbf{y}\right)\right] = \int g\left(\mathbf{y}\right) p\left(\mathbf{y}; \mathbf{C}^{ij}\mathbf{m}^{ij}, \mathbf{C}^{ij}\right) d\mathbf{y}, \tag{B.3a}$$

then

$$[\mathbf{MS}]_{i,j} = \mathbb{E}_{\mathbf{y};\boldsymbol{\theta}^0}\left[\frac{p\left(\mathbf{y};\boldsymbol{\theta}^i\right)}{p\left(\mathbf{y};\boldsymbol{\theta}^0\right)}\frac{p\left(\mathbf{y};\boldsymbol{\theta}^j\right)}{p\left(\mathbf{y};\boldsymbol{\theta}^0\right)}\right] = [\mathbf{MS}]_{i,j}\int p\left(\mathbf{y}; \mathbf{C}^{ij}\mathbf{m}^{ij}, \mathbf{C}^{ij}\right) d\mathbf{y} \tag{B.3b}$$

$$\mathbf{h}\left(\boldsymbol{\theta}^i, \boldsymbol{\theta}^j\right) = \mathbb{E}_{\mathbf{y};\boldsymbol{\theta}^0}\left[\frac{\partial \ln p\left(\mathbf{y};\boldsymbol{\theta}^i\right)}{\partial \boldsymbol{\theta}}\frac{p\left(\mathbf{y};\boldsymbol{\theta}^i\right)}{p\left(\mathbf{y};\boldsymbol{\theta}^0\right)}\frac{p\left(\mathbf{y};\boldsymbol{\theta}^j\right)}{p\left(\mathbf{y};\boldsymbol{\theta}^0\right)}\right] = [\mathbf{MS}]_{i,j}\ \mathbb{E}_{\mathbf{y}}^{ij}\left[\frac{\partial \ln p\left(\mathbf{y};\boldsymbol{\theta}^i\right)}{\partial \boldsymbol{\theta}}\right] \tag{B.3c}$$

Therefore in the following we consider the representation $\mathbf{y} \sim \mathcal{N}_M\left(\mathbf{C}^{ij}\mathbf{m}^{ij}, \mathbf{C}^{ij}\right)$, where $\mathbf{C}^{ij}$, $\mathbf{m}^{ij}$ are given by (B.2a-B.2d). To compute the missing expectations, let us recall that $p\left(\mathbf{y};\boldsymbol{\theta}\right) = e^{-\frac{1}{2}\phi(\mathbf{y};\boldsymbol{\theta})} / \left(\sqrt{2\pi}^M \sqrt{\left|\mathbf{C}\left(\boldsymbol{\theta}\right)\right|}\right)$ where $\phi\left(\mathbf{y};\boldsymbol{\theta}\right) = tr\left(\mathbf{C}\left(\boldsymbol{\theta}\right)^{-1}\widehat{\mathbf{C}\left(\boldsymbol{\theta}\right)}\right)$ and $\widehat{\mathbf{C}\left(\boldsymbol{\theta}\right)} = \left(\mathbf{y} - \mathbf{m}\left(\boldsymbol{\theta}\right)\right)\left(\mathbf{y} - \mathbf{m}\left(\boldsymbol{\theta}\right)\right)^\top$ and that

$$\frac{\partial \mathbf{C}\left(\boldsymbol{\theta}\right)^{-1}}{\partial \theta_k} = -\mathbf{C}\left(\boldsymbol{\theta}\right)^{-1}\frac{\partial \mathbf{C}\left(\boldsymbol{\theta}\right)}{\partial \theta_k}\mathbf{C}\left(\boldsymbol{\theta}\right)^{-1},$$

$$\frac{\partial \ln\left|\mathbf{C}\left(\boldsymbol{\theta}\right)\right|}{\partial \theta_k} = -tr\left(\frac{\partial \mathbf{C}\left(\boldsymbol{\theta}\right)^{-1}}{\partial \theta_k}\mathbf{C}\left(\boldsymbol{\theta}\right)\right).$$

Then,

$$\frac{\partial \phi\left(\mathbf{y};\boldsymbol{\theta}\right)}{\partial \theta_k} = -2\frac{\partial \mathbf{m}\left(\boldsymbol{\theta}\right)}{\partial \theta_k}^\top \mathbf{C}\left(\boldsymbol{\theta}\right)^{-1}\left(\mathbf{y} - \mathbf{m}\left(\boldsymbol{\theta}\right)\right) + \left(\mathbf{y} - \mathbf{m}\left(\boldsymbol{\theta}\right)\right)^\top \times \frac{\partial \mathbf{C}\left(\boldsymbol{\theta}\right)^{-1}}{\partial \theta_k}\left(\mathbf{y} - \mathbf{m}\left(\boldsymbol{\theta}\right)\right), \tag{B.4}$$

$$\frac{\partial \ln p\left(\mathbf{y};\boldsymbol{\theta}\right)}{\partial \theta_k} = \frac{1}{2}tr\left(\frac{\partial \mathbf{C}\left(\boldsymbol{\theta}\right)^{-1}}{\partial \theta_k}\left(\mathbf{C}\left(\boldsymbol{\theta}\right) - \widehat{\mathbf{C}\left(\boldsymbol{\theta}\right)}\right)\right) + \frac{\partial \mathbf{m}\left(\boldsymbol{\theta}\right)}{\partial \theta_k}^\top \mathbf{C}\left(\boldsymbol{\theta}\right)^{-1}\left(\mathbf{y} - \mathbf{m}\left(\boldsymbol{\theta}\right)\right). \tag{B.5}$$

From (B.5), we have that

$$\mathbb{E}_{\mathbf{y}}^{ij}\left[\frac{\partial \ln p\left(\mathbf{y};\boldsymbol{\theta}\right)}{\partial \theta_k}\right] = \frac{1}{2}tr\left(\frac{\partial \mathbf{C}\left(\boldsymbol{\theta}\right)^{-1}}{\partial \theta_k}\left(\mathbf{C}\left(\boldsymbol{\theta}\right) - \mathbb{E}_{\mathbf{y}}^{ij}\left[\widehat{\mathbf{C}\left(\boldsymbol{\theta}\right)}\right]\right)\right) + \frac{\partial \mathbf{m}\left(\boldsymbol{\theta}\right)}{\partial \theta_k}^\top \mathbf{C}\left(\boldsymbol{\theta}\right)^{-1}\left(\mathbb{E}_{\mathbf{y}}^{ij}\left[\mathbf{y}\right] - \mathbf{m}\left(\boldsymbol{\theta}\right)\right), \tag{B.6}$$

where

$$\mathbb{E}_{\mathbf{y}}^{ij}\left[\widehat{\mathbf{C}\left(\boldsymbol{\theta}\right)}\right] = \mathbf{C}^{ij} + \left(\mathbb{E}_{\mathbf{y}}^{ij}\left[\mathbf{y}\right] - \mathbf{m}\left(\boldsymbol{\theta}\right)\right)\left(\mathbb{E}_{\mathbf{y}}^{ij}\left[\mathbf{y}\right] - \mathbf{m}\left(\boldsymbol{\theta}\right)\right)^\top. \tag{B.7}$$

Finally,

$$[\mathbf{MS}]_{i,j} = \sqrt{\frac{\left|\mathbf{C}^{ij}\right|\left|\mathbf{C}\left(\boldsymbol{\theta}^0\right)\right|}{\left|\mathbf{C}\left(\boldsymbol{\theta}^i\right)\right|\left|\mathbf{C}\left(\boldsymbol{\theta}^j\right)\right|}}\, e^{\frac{1}{2}\left[\left(\mathbf{m}^{ij}\right)^\top \mathbf{C}^{ij}\mathbf{m}^{ij} - \delta^{ij}\right]}, \tag{B.8a}$$

$$\left[\mathbf{h}\left(\boldsymbol{\theta}^i, \boldsymbol{\theta}^j\right)\right]_k = [\mathbf{MS}]_{i,j}\left[\boldsymbol{\alpha}\left(\boldsymbol{\theta}^i\right)\right]_k, \tag{B.8b}$$

where

$$\left[\boldsymbol{\alpha}\left(\boldsymbol{\theta}^i\right)\right]_k = \begin{pmatrix} \frac{1}{2}tr\left(\frac{\partial \mathbf{C}\left(\boldsymbol{\theta}^i\right)^{-1}}{\partial \theta_k}\left(\mathbf{C}\left(\boldsymbol{\theta}^i\right) - \mathbf{C}^{ij}\right) - \frac{\partial \mathbf{C}\left(\boldsymbol{\theta}^i\right)^{-1}}{\partial \theta_k}\right. \\ \left.\times\left(\mathbf{C}^{ij}\mathbf{m}^{ij} - \mathbf{m}\left(\boldsymbol{\theta}^i\right)\right)\left(\mathbf{C}^{ij}\mathbf{m}^{ij} - \mathbf{m}\left(\boldsymbol{\theta}^i\right)\right)^\top\right) \\ + \frac{\partial \mathbf{m}\left(\boldsymbol{\theta}^i\right)}{\partial \theta_k}^\top \mathbf{C}\left(\boldsymbol{\theta}^i\right)^{-1}\left(\mathbf{C}^{ij}\mathbf{m}^{ij} - \mathbf{m}\left(\boldsymbol{\theta}^i\right)\right) \end{pmatrix}.$$

Moreover, since $\mathbf{C}^{0j} = \mathbf{C}\left(\boldsymbol{\theta}^j\right)$, $\mathbf{m}^{0j} = \mathbf{C}\left(\boldsymbol{\theta}^j\right)^{-1}\mathbf{m}\left(\boldsymbol{\theta}^j\right)$, $\mathbf{C}^{0j}\mathbf{m}^{0j} = \mathbf{m}\left(\boldsymbol{\theta}^j\right)$ and $\delta^{0j} = \mathbf{m}\left(\boldsymbol{\theta}^j\right)^\top \mathbf{C}\left(\boldsymbol{\theta}^j\right)^{-1}\mathbf{m}\left(\boldsymbol{\theta}^j\right)$, then $[\mathbf{MS}]_{0,j} = 1$, and $\mathbf{h}\left(\boldsymbol{\theta}^0, \boldsymbol{\theta}^j\right) = \boldsymbol{\alpha}\left(\boldsymbol{\theta}^i\right)$.

Acknowledgments

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