Bird's-eye view on our work

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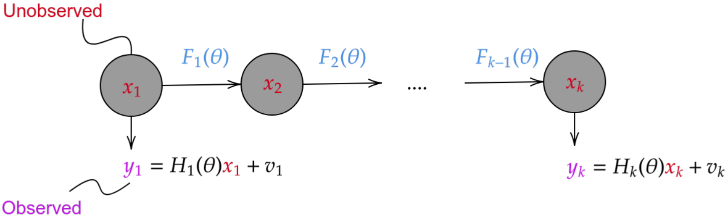
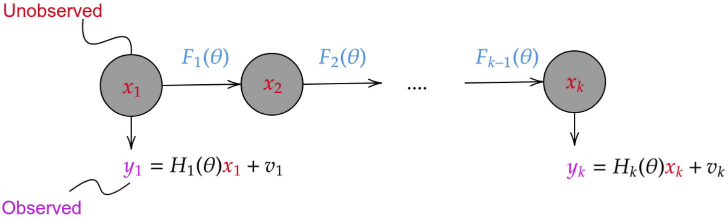
I. On Hybrid Estimation for Linear Discrete State-Space Models

Signal Model



$$\begin{cases} \mathbf{x}_k &= \mathbf{F}_{k-1}(\boldsymbol{\theta})\,\mathbf{x}_{k-1}, \\ \mathbf{y}_k &= \mathbf{H}_k(\boldsymbol{\theta})\,\mathbf{x}_k + \mathbf{v}_k, \end{cases}$$

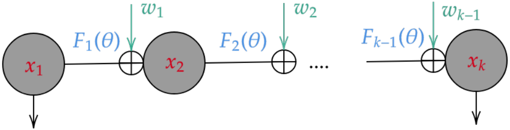
Signal Model



$$\begin{cases} \mathbf{x}_k &= \mathbf{F}_{k-1}(\boldsymbol{\theta})\,\mathbf{x}_{k-1}, \\ \mathbf{y}_k &= \mathbf{H}_k(\boldsymbol{\theta})\,\mathbf{x}_k + \mathbf{v}_k, \end{cases}$$

$$\begin{pmatrix} \mathbf{y}_1 \\ \vdots \\ \mathbf{y}_k \end{pmatrix} = \begin{pmatrix} \mathbf{H'}_1(\theta) \\ \vdots \\ \mathbf{H'}_k(\theta) \end{pmatrix} \mathbf{x}_1 + \begin{pmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_k \end{pmatrix}$$

$$\boxed{\overline{\mathbf{y}}_{1:k} = \overline{\mathbf{H'}}_{1:k}(\theta)\mathbf{x}_1 + \overline{\mathbf{v}}_{1:k}}, \ \overline{\mathbf{y}}_{1:k} \underset{i.i.d}{\sim} \mathcal{CN}(\overline{\mathbf{H'}}_{1:k}(\theta)x_1, \mathbf{C}_{\overline{\mathbf{v}}_{1:k}})$$



$$\begin{cases} \mathbf{x}_k & = \mathbf{F}_{k-1}(\boldsymbol{\theta})\,\mathbf{x}_{k-1} + \mathbf{w}_{k-1}, \\ \mathbf{y}_k & = \mathbf{H}_k(\boldsymbol{\theta})\,\mathbf{x}_k + \mathbf{v}_k, \end{cases}$$

$$\begin{pmatrix} \mathbf{y}_1 \\ \vdots \\ \mathbf{y}_k \end{pmatrix} = \begin{pmatrix} \mathbf{A}_1(\boldsymbol{\theta}) \\ \vdots \\ \mathbf{A}_k(\boldsymbol{\theta}) \end{pmatrix} \mathbf{x}_1 + \begin{pmatrix} \mathbf{n}_1 \\ \vdots \\ \mathbf{n}_k(\boldsymbol{\theta}) \end{pmatrix}$$

$$\boxed{\overline{\mathbf{y}}_k = \overline{\mathbf{A}}_{1:k}(\boldsymbol{\theta})\mathbf{x}_1 + \overline{\mathbf{n}}_{1:k}(\boldsymbol{\theta})},\ \overline{\mathbf{y}}_{1:k} \underset{n.i.d}{\sim} \mathcal{CN}(\overline{\mathbf{A}}_{1:k}(\theta)\mathbf{x}_1, \mathbf{C}_{\overline{\mathbf{n}}_{1:k}(\boldsymbol{\theta})})$$

General Signal Model

Consider the general LDSS model,

$$\begin{cases} \mathbf{x}_l & = \mathbf{F}_{l-1}(\boldsymbol{\theta})\,\mathbf{x}_{l-1} + \mathbf{w}_{l-1}, \quad 2 \leq l \leq k, \\ \mathbf{y}_l & = \mathbf{H}_l(\boldsymbol{\theta})\,\mathbf{x}_l + \mathbf{v}_l, \qquad 1 \leq l \leq k. \\ & \Downarrow \end{cases} \tag{1}$$

$$\boxed{\overline{\mathbf{y}}_{1:k} = \overline{\mathbf{A}}_{1:k}(\boldsymbol{\theta})\,\mathbf{x}_1 + \overline{\mathbf{n}}_{\mathbf{1}:\boldsymbol{k}}(\boldsymbol{\theta})}, \mathbf{x}_1 \sim \mathcal{CN}(\mathbf{0}, \mathbf{C}_{\mathbf{x}_1})$$

Twofold Goal:

- How to solve the hybrid estimation of $(\mathbf{x}_1, \boldsymbol{\theta})$ recursively ?
- How to evaluate the asymptotic performance of the estimator recursively ?

Joint MAP-MLE

$$(\widehat{\mathbf{x}}_{\mathrm{MAP}}; \widehat{\boldsymbol{\theta}}_{\mathrm{ML}}) \stackrel{def}{=} \arg\max_{\mathbf{x}_1, \boldsymbol{\theta}} p(\overline{\mathbf{y}}_{1:k}, \mathbf{x}_1; \boldsymbol{\theta}),$$

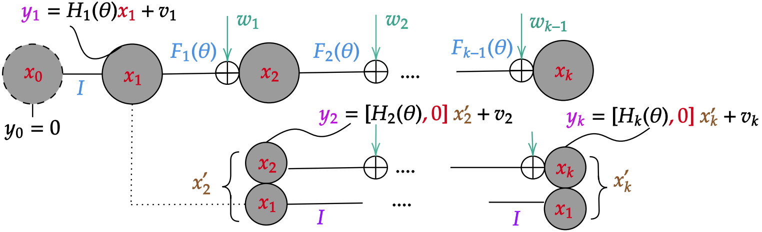
- $\widehat{\mathbf{x}}_{\mathrm{MAP}} \stackrel{def}{=} \widehat{\mathbf{x}}_{1|k}(\boldsymbol{\theta})$: MAP estimate of $\mathbf{x}_1$
- $\widehat{\boldsymbol{\theta}}_{\mathrm{ML}} \stackrel{def}{=} \widehat{\boldsymbol{\theta}}$: MLE estimate of $\boldsymbol{\theta}$

$$\arg\max_{\boldsymbol{\theta}} \mathcal{L}(\overline{\mathbf{y}}_{1:k}; \widehat{\mathbf{x}}_{1|k}(\boldsymbol{\theta}), \boldsymbol{\theta}) = \arg\max_{\boldsymbol{\theta}} \{N_k(\boldsymbol{\theta}) - M_k(\boldsymbol{\theta})\},$$

Why recursively? $\mathbf{C}^{-1}_{\overline{\mathbf{n}}_{1:k}}(\boldsymbol{\theta})$, $|\mathbf{C}_{\overline{\mathbf{n}}_{1:k}}(\boldsymbol{\theta})|$

Sketch of the Recursive Estimation

1. Recursive estimation of $\widehat{\mathbf{x}}_{1|k}(\boldsymbol{\theta})$



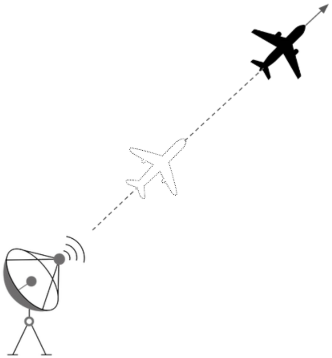
Fixed-point smoother

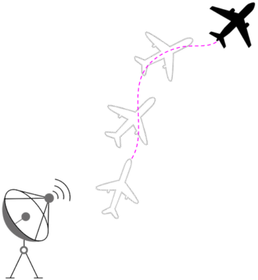
2. $\mathcal{N}_k(\boldsymbol{\theta})$: Kalman filter on the augmented system.
3. $\mathcal{M}_k(\boldsymbol{\theta})$: Kalman Filter on the original system.

An illustrative Example: RADAR Application

- Emission: $e_T(t)\exp^{j2\pi f_c t}$
- $\mathbf{y}_1 = \mathbf{h}(\theta)\beta x_1 + \mathbf{v}_1 \in \mathbb{C}^N$
- x_1: Complex Back-scattering coefficient
- $\mathbf{h}(\theta) = \left(1, \exp^{j2\pi\theta}, \dots, \exp^{j2\pi(N-1)\theta}\right)^T$
- $\theta = \frac{-2vTf_c}{c}$: Normalized Doppler frequency

- $x_1 \propto \sqrt{RCS}$
- RCS : Radar Cross Section





Swerling 1 ($|x_1|^2 \sim \chi_2$): $x_1 \sim \mathcal{CN}(0, \sigma^2_{x_1})$
Partially coherent signals.

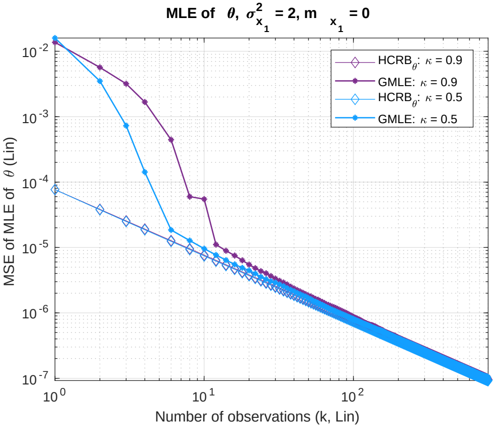
$$x_k = f_{k-1} x_{k-1} + w_{k-1}, w_{k-1} \underset{i.i.d}{\sim} \mathcal{CN}(0, \sigma^2_{w_{k-1}})$$

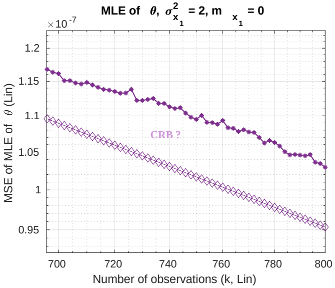
$$\mathbf{y}_k = \mathbf{h}_k(\theta) x_k + v_k, v_k \underset{i.i.d}{\sim} \mathcal{CN}(\mathbf{0}, \mathbf{C}_{\mathbf{v}_k})$$

$$\boxed{\overline{\mathbf{y}}_{1:k} = \overline{\mathbf{a}}_{1:k}(\theta) x_1 + \overline{\mathbf{n}}_{1:k}(\theta)},$$

$$\overline{\mathbf{y}}_{1:k} \underset{n.i.d}{\sim} \mathcal{CN}(\overline{\mathbf{a}}_{1:k}(\theta), \mathbf{C}_{\overline{\mathbf{n}}_{1:k}}(\theta))$$

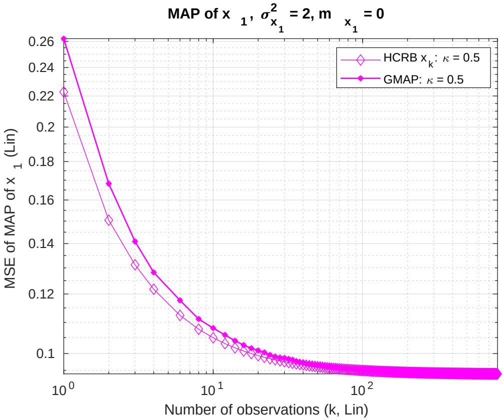
MLE of θ

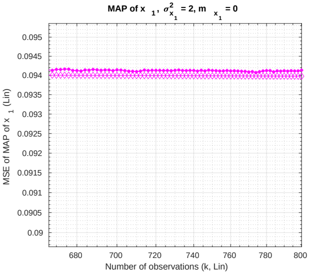




Consistent but (efficient ??)

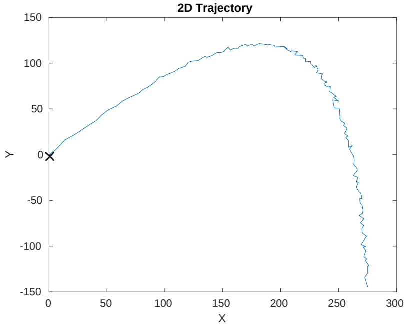
MAPMLE of $\widehat{x}_{1|k}(\hat{\theta})$





Efficient but not consistent

An illustrative example: Projectile motion



Joint estimation of $(\mathbf{z}_k, \mathbf{z}_1, \rho)$,$\mathbf{z}_1 \sim \mathcal{N}(\mathbf{0}, \mathbf{C}_{\mathbf{z}_1})$

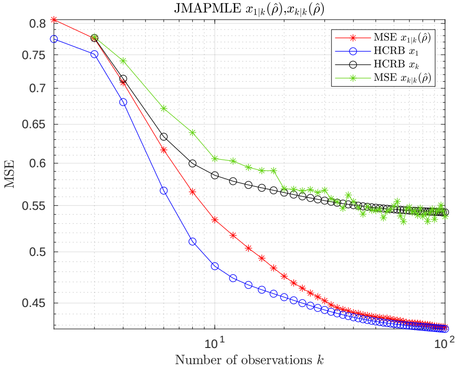
$$F_c = -\rho \mathbf{v} + \mathbf{w}, F_g = -m\mathbf{g}$$

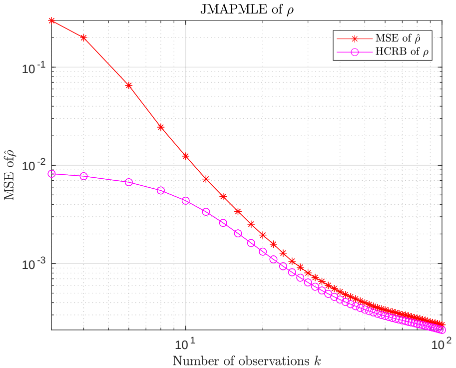
$$\mathbf{z}_k = \begin{bmatrix} x_k \\ y_k \\ v_k^x \\ v_k^y \end{bmatrix},$$

$$\mathbf{z}_k = \mathbf{F}_{k-1}(\rho)\mathbf{z}_{k-1} + \mathbf{g}_{k-1} + \mathbf{w}_{k-1}$$

$$\mathbf{p}_k = \mathbf{H}_k \mathbf{z}_k + \mathbf{v}_k$$

MAPMLE of $\widehat{\mathbf{z}}_{1|k}(\rho)$



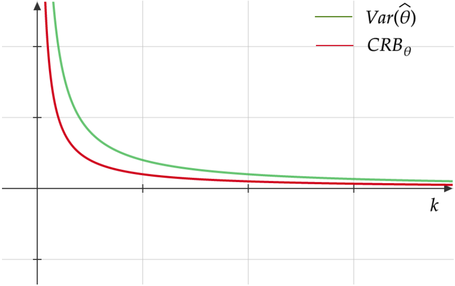


<u>Efficient but not consistent</u>

II. On Hybrid and Modified Cramér-Rao Bounds for MDS

Cramér-Rao Bound

$$\mathcal{L}(\theta) = \ln p(\mathbf{y}|\theta)$$



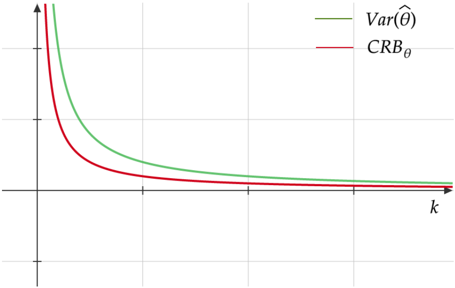
If $\mathbb{E}[\hat{\theta}] = \theta($ wide-sense unbiased$)$

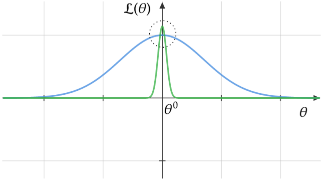
$\mathrm{Var}(\hat{\theta}) \geq \mathrm{CRB}_\theta$

$$\mathrm{CRB}_\theta = \mathbb{E}\left[-\frac{\partial^2 \mathcal{L}(\theta)}{\partial^2 \theta}\right]^{-1} = I(\theta)^{-1}$$

Cramér-Rao Bound

$$\mathcal{L}(\theta) = \ln p(\mathbf{y}|\theta)$$





If $\mathbb{E}[\hat{\theta}] = \theta$(wide-sense unbiased)

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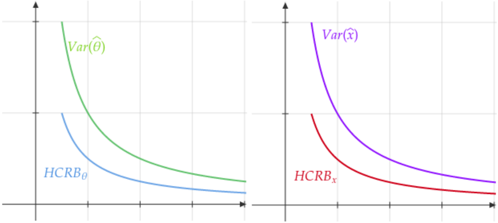
$$\mathrm{CRB}_\theta = \mathbb{E}\left[-\frac{\partial^2 \mathcal{L}(\theta)}{\partial^2 \theta}\right]^{-1} = I(\theta)^{-1}$$

$$I(\theta) \geq I(\theta)$$

Hybrid Cramér-Rao Bound

$$\mathcal{L} = \ln p(\mathbf{y}, x|\theta)$$



$$\text{If } \mathbb{E}_{\mathbf{y},x|\theta}\begin{bmatrix} \hat{x} - x \\ \hat{\theta} - \theta \end{bmatrix} = \begin{bmatrix} \mu(\theta) \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} Var(\hat{x}^2) & Cov(\hat{x}, \hat{\theta}) \\ . & Var(\hat{\theta}^2) \end{bmatrix} \succeq \mathbb{E}_{\mathbf{y},x|\theta}\begin{bmatrix} -\frac{\partial^2 \mathcal{L}}{\partial^2 x} & -\frac{\partial^2 \mathcal{L}}{\partial x \partial \theta} \\ . & -\frac{\partial^2 \mathcal{L}}{\partial^2 \theta} \end{bmatrix}^{-1} = \begin{bmatrix} HCRB_x & * \\ * & HCRB_\theta \end{bmatrix}$$

Hybrid Cramér-Rao Bound for MDS

$$\begin{cases} \mathbf{x}_k = \mathbf{f}_{k-1}\left(\mathbf{x}_{k-1}, \mathbf{w}_{k-1}(\boldsymbol{\beta}), \boldsymbol{\alpha}\right), \\ \mathbf{y}_k = \mathbf{h}_k\left(\mathbf{x}_k, \mathbf{v}_k(\boldsymbol{\mu}), \boldsymbol{\lambda}\right), \end{cases}$$

$$\boldsymbol{\theta}^\top = (\boldsymbol{\beta}^\top, \boldsymbol{\alpha}^\top, \boldsymbol{\mu}^\top, \boldsymbol{\lambda}^\top),$$

$$\mathbf{J}_{\mathbf{x}_k, \boldsymbol{\theta}} = \mathbb{E}\left[-\frac{\partial^2 \ln p(\overline{\mathbf{y}}_{1:k}, \mathbf{x}_k | \boldsymbol{\theta})}{\partial({\mathbf{x}_k}^\top, \boldsymbol{\theta}^\top)^\top \partial({\mathbf{x}_k}^\top, \boldsymbol{\theta}^\top)}\right]$$

Hybrid Cramér-Rao Bound for MDS

$$\begin{cases} \mathbf{x}_k = \mathbf{f}_{k-1}\left(\mathbf{x}_{k-1}, \mathbf{w}_{k-1}(\boldsymbol{\beta}), \boldsymbol{\alpha}\right), \\ \mathbf{y}_k = \mathbf{h}_k\left(\mathbf{x}_k, \mathbf{v}_k(\boldsymbol{\mu}), \boldsymbol{\lambda}\right), \end{cases}$$

$$\boldsymbol{\theta}^\top = (\boldsymbol{\beta}^\top, \boldsymbol{\alpha}^\top, \boldsymbol{\mu}^\top, \boldsymbol{\lambda}^\top),$$

$$\mathbf{J}_{\mathbf{x}_k,\boldsymbol{\theta}} = \mathbb{E}\left[-\frac{\partial^2 \ln p(\overline{\mathbf{y}}_{1:k}, \mathbf{x}_k|\boldsymbol{\theta})}{\partial({\mathbf{x}_k}^\top, \boldsymbol{\theta}^\top)^\top \partial({\mathbf{x}_k}^\top, \boldsymbol{\theta}^\top)}\right] \Rightarrow \mathbf{J}_{\overline{\mathbf{x}}_{1:k},\boldsymbol{\theta}} = \mathbb{E}\left[-\frac{\partial^2 \ln p(\overline{\mathbf{y}}_k, \overline{\mathbf{x}}_{1:k}|\boldsymbol{\theta})}{\partial({\overline{\mathbf{x}}_{1:k}}^\top, \boldsymbol{\theta}^\top)^\top \partial({\overline{\mathbf{x}}_{1:k}}^\top, \boldsymbol{\theta}^\top)}\right]$$

$$\Downarrow$$

$$= \mathbb{E}\begin{pmatrix} \Delta_{\overline{\mathbf{x}}_{2:k-1}}^{\overline{\mathbf{x}}_{2:k-1}} & \Delta_{\overline{\mathbf{x}}_{2:k-1}}^{\mathbf{x}_k} & \Delta_{\overline{\mathbf{x}}_{2:k-1}}^{\boldsymbol{\theta}} \\ (.)^\top & \Delta_{\mathbf{x}_k}^{\mathbf{x}_k} & \Delta_{\mathbf{x}_k}^{\boldsymbol{\theta}} \\ (.)^\top & (.)^\top & \Delta_{\boldsymbol{\theta}}^{\boldsymbol{\theta}} \end{pmatrix} \mathcal{L}_k$$

Hybrid Cramér-Rao Bound for MDS

$$\begin{cases} \mathbf{x}_k = \mathbf{f}_{k-1}\left(\mathbf{x}_{k-1}, \mathbf{w}_{k-1}(\boldsymbol{\beta}), \boldsymbol{\alpha}\right), \\ \mathbf{y}_k = \mathbf{h}_k\left(\mathbf{x}_k, \mathbf{v}_k(\boldsymbol{\mu}), \boldsymbol{\lambda}\right), \end{cases}$$

$$\boldsymbol{\theta}^\top = (\boldsymbol{\beta}^\top, \boldsymbol{\alpha}^\top, \boldsymbol{\mu}^\top, \boldsymbol{\lambda}^\top),$$

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$$\Downarrow$$

$$= \mathbb{E}\begin{pmatrix} \Delta_{\overline{\mathbf{x}}_{2:k-1}}^{\overline{\mathbf{x}}_{2:k-1}} & \Delta_{\overline{\mathbf{x}}_{2:k-1}}^{\mathbf{x}_k} & \Delta_{\overline{\mathbf{x}}_{2:k-1}}^{\boldsymbol{\theta}} \\ (.)^\top & \Delta_{\mathbf{x}_k}^{\mathbf{x}_k} & \Delta_{\mathbf{x}_k}^{\boldsymbol{\theta}} \\ (.)^\top & (.)^\top & \Delta_{\boldsymbol{\theta}}^{\boldsymbol{\theta}} \end{pmatrix} \mathcal{L}_k$$

$$\mathbf{J} \succeq \mathbf{J}_{\mathbf{x}_k,\boldsymbol{\theta}} \Leftrightarrow \mathrm{HCRB}_{\mathbf{x}_k,\boldsymbol{\theta}} \succeq \mathbf{J}^{-1}$$

Modified Cramér-Rao Bound

$$\mathbf{y} = f(\boldsymbol{\theta}, \mathbf{v}), p(\mathbf{y}|\boldsymbol{\theta})$$

$$\mathbf{C}_{\widehat{\boldsymbol{\theta}}} \geq \mathbb{E}\left[-\frac{\partial^2 \ln p(\mathbf{y}|\boldsymbol{\theta})}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}^\top}\right]^{-1}$$

Modified Cramér-Rao Bound

$$\mathbf{y} = f(\boldsymbol{\theta}, \mathbf{v}), p(\mathbf{y}|\boldsymbol{\theta})$$

$$\mathbf{C}_{\widehat{\boldsymbol{\theta}}} \geq \mathbb{E}\left[-\frac{\partial^2 \ln p(\mathbf{y}|\boldsymbol{\theta})}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}^\top}\right]^{-1}$$

$$\mathbf{y} = f(\boldsymbol{\theta}, \mathbf{x}, \mathbf{v}), \mathbf{x} \sim p(\mathbf{x})$$

$$p(\mathbf{y}|\boldsymbol{\theta}) = \int p(\mathbf{y}, \mathbf{x}|\boldsymbol{\theta}) d\mathbf{x}$$

$$\Downarrow$$

$$MCRB_{\boldsymbol{\theta}} = \mathbb{E}\left[-\frac{\partial^2 \ln p(\mathbf{y}, \mathbf{x}|\boldsymbol{\theta})}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}^\top}\right]^{-1}$$

Modified Cramér-Rao Bound

$$\mathbf{y} = f(\boldsymbol{\theta}, \mathbf{v}), p(\mathbf{y}|\boldsymbol{\theta})$$

$$\mathbf{C}_{\widehat{\boldsymbol{\theta}}} \geq \mathbb{E}\left[-\frac{\partial^2 \ln p(\mathbf{y}|\boldsymbol{\theta})}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}^\top}\right]^{-1}$$

$$\mathbf{y} = f(\boldsymbol{\theta}, {\color{red}\mathbf{x}}, \mathbf{v}), \mathbf{x} \sim p(\mathbf{x})$$

$$p(\mathbf{y}|\boldsymbol{\theta}) = \int p(\mathbf{y}, \mathbf{x}|\boldsymbol{\theta}) d\mathbf{x}$$

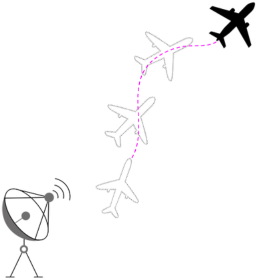
$$\Downarrow$$

$$MCRB_{\boldsymbol{\theta}} = \mathbb{E}\left[-\frac{\partial^2 \ln p(\mathbf{y}, \mathbf{x}|\boldsymbol{\theta})}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}^\top}\right]^{-1}$$

$$\Downarrow \text{+Constraint}$$

$$\mathbf{J}_{\mathbf{x},\boldsymbol{\theta}}^{-1} = \mathbb{E}\left[\begin{array}{c|c} * & * \\ * & {\color{red}\overline{MCRB}_{\boldsymbol{\theta}}} \end{array}\right]$$

$$CRB_{\boldsymbol{\theta}} \geq {\color{red}\overline{MCRB}_{\boldsymbol{\theta}}} \geq MCRB_{\boldsymbol{\theta}}$$



x_1 unknown deterministic
Partially coherent signals.

$$x_k = \kappa x_{k-1} + w_{k-1}, w_{k-1} \underset{i.i.d}{\sim} \mathcal{CN}(0, \sigma^2_{w_{k-1}})$$

$$\mathbf{y}_k = \mathbf{h}_k(\nu) x_k + v_k, v_k \underset{i.i.d}{\sim} \mathcal{CN}(\mathbf{0}, \mathbf{C}_{\mathbf{v}_k})$$

$$\boxed{\overline{\mathbf{y}}_{1:k} = \overline{\mathbf{a}}_{1:k}(\nu) x_1 + \overline{\mathbf{n}}_{1:k}(\nu)},$$
$$\overline{\mathbf{y}}_{1:k} \underset{n.i.d}{\sim} \mathcal{CN}(\overline{\mathbf{a}}_{1:k}(\theta), \mathbf{C}_{\overline{\mathbf{n}}_{1:k}}(\theta))$$

Goal:

- How to evaluate the asymptotic performance of the estimator of $\boldsymbol{\theta} = (x_1, \nu)^\top$ recursively ?

Comparison

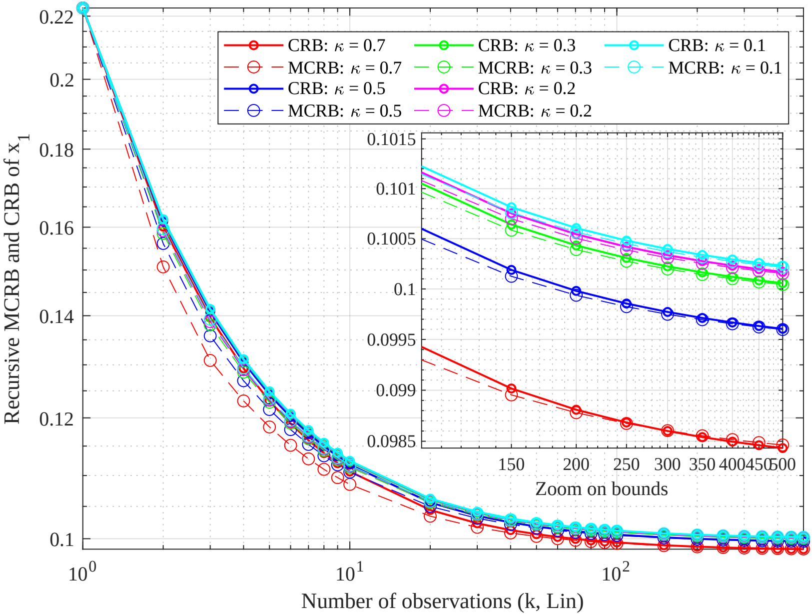
- Method 1. (Slepian-Bangs formula)

$$J_{i,j} = 2\Re e\left(\frac{\partial \boldsymbol{\mu}}{\partial \theta_i}^H \mathbf{C}^{-1} \frac{\partial \boldsymbol{\mu}}{\partial \theta_j}\right) + Tr\left(\mathbf{C}^{-1}\frac{\partial \mathbf{C}^{-1}}{\partial \theta_i}\mathbf{C}^{-1}\frac{\partial \mathbf{C}^{-1}}{\partial \theta_j}\right)$$

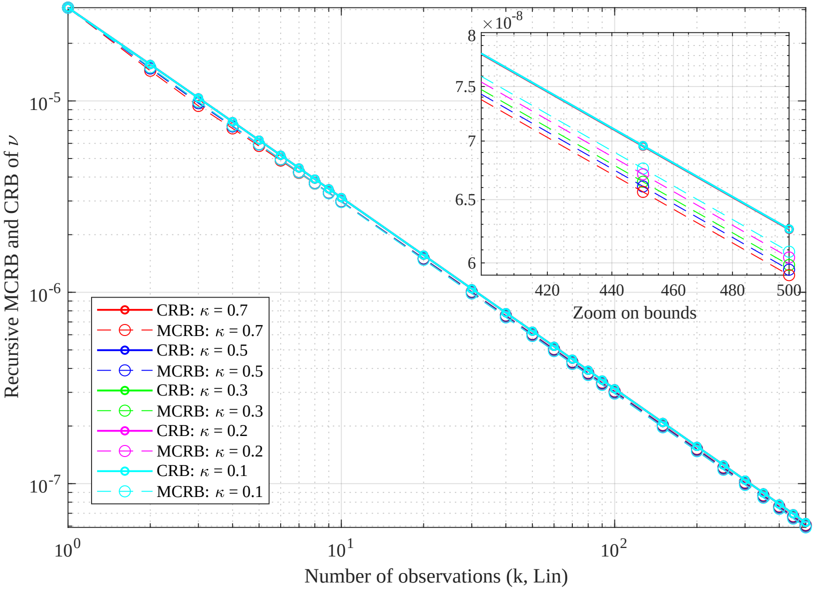
- Method 2. Recursive $\overline{MCRB}_{\boldsymbol{\theta}}$

$$\mathbf{J}_{\overline{\mathbf{x}}_{2:k},\boldsymbol{\theta}} = \begin{pmatrix} \Delta_{\overline{\mathbf{x}}_{2:k}}^{\overline{\mathbf{x}}_{2:k}} & \Delta_{\overline{\mathbf{x}}_{2:k}}^{\mathbf{x}_1} & \Delta_{\overline{\mathbf{x}}_{2:k}}^{\boldsymbol{\theta}} \\ & \Delta_{x_1}^{x_1} & \Delta_{x_1}^{\nu} \\ & & \Delta_{\nu}^{\nu} \end{pmatrix}.$$

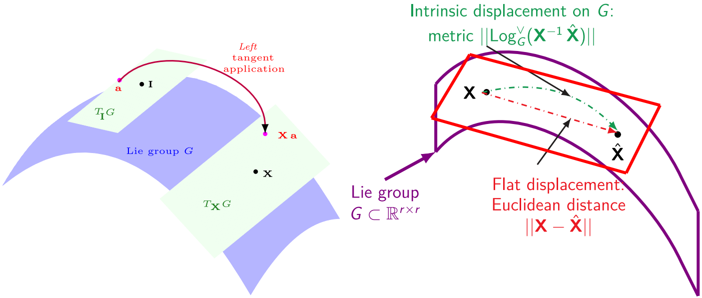
Comparison between the CRB and MCRB



Comparison between the CRB and MCRB



III. Modified Cramér-Rao Bound on Lie Groups



Images are courtesy of S. Labsir

Goal: Derive an *intrinsic* $\mathbf{MCRB}_{\mathbf{X}}$ =?

- Statistical Mean ?
- Bias ?
- MSE ?

Illustrative Example