Robust semiparametric efficient estimator for time delay and Doppler estimation

Lorenzo Ortega, Stefano Fortunati

Robust semiparametric efficient estimator for time delay and Doppler estimation

Member, IEEE, Lorenzo Ortega, *Member, IEEE*, Stefano Fortunati.

***Abstract*—This paper explores time-delay and Doppler estimation in the presence of unknown heavy-tailed disturbance. Conventional methods for achieving optimal mean squared error performance rely on the maximum likelihood estimator (MLE), which is consistent and asymptotically efficient under the unrealistic assumption of a perfect a-priori knowledge of the noise distribution. However, in practical situations, the noise distribution is often unknown, and classical parametric estimation procedures are no longer able to guarantee the statistical efficiency. In this work, by relying on the semiparametric theory, we present an original *rank-based* and *distribution-free* R-estimator which have the remarkable property to be *parametrically efficient*, i.e. it attains the "classical" Cramér-Rao Bound, *irrespective of the unknown noise distribution*, provided that the latter belongs to the family of Complex Elliptically Simmetric (CES) distributions.**

***Index Terms*—Semiparametric models, Robust time-delay and Doppler estimation, band-limited signals.**

I. Introduction

Time-delay and Doppler estimation are fundamental operations across numerous engineering domains, including communications, radar or navigation systems [1]–[10], as they form the initial task of the receiver [5], [8], [9]. Given its significance, it is highly valuable to establish the best achievable estimation performance in terms of mean squared error (MSE). Under the standard parametric assumption, the Cramér-Rao bound (CRB) [11], [12] serves this purpose. Moreover, it is well known that the CRB is asymptotically achieved by the MLE (at least under some regularity condition on the data pdf) [13]. Therefore, several CRB formulations for time-delay and Doppler estimation have been derived over the years, for narrow-band and wide-band signals [2], [14]–[23]. Additionally, recent studies have examined scenarios where the actual signal model differs from the assumed one at the receiver. In [24]–[28], it is assumed that the signal model differs from the true one due to phenomena such as multipath, interference or high receiver dynamics. Also, in [29], [30], the authors consider the case in which the true signal model is characterized by a non-Gaussian, heavy-tailed CES distribution while the assumed one is the standard complex normal distribution. In both types of studies, the theory of model misspecification is adopted [31], [32], that is the estimation performance are characterized in terms of pseudo-true parameters and the Misspecified CRB (MCRB). Specifically, the MCRB yields the error covariance matrix for the MLE when the model is misspecified, known as the Misspecified MLE (MMLE) [31, Theo. 2], [32, Sec. 4.4.3]. Using this approach, one can show that, when the noise CES distribution is misspecified, the MMLE for time-delay and Doppler estimation is $\sqrt{N}$-consistent with respect to the true parameters as the number N of observations goes to infinity. This is a key finding, indicating that the MMLE, originally developed under the assumption of Gaussian data, can still be applied when the noise follows a non-Gaussian, heavy-tailed CES distribution. Nevertheless, along with the consistency, we would like to have estimators achieving asymptotic efficiency with respect to the true and unknown noise distribution. To address this, one must consider the theory of semiparametric estimation, recently revisited in [33]–[36] for the family of CES distributions. The semiparametric framework allows us to derive a lower bound, known as the Semiparametric CRB (SCRB), representing the lowest MSE achievable by any consistent estimator in the presence of an unspecified CES distribution. Furthermore, for our time-delay and Doppler estimation problem, from the results in [34], it can be readily shown that the SCRB equates the CRB of the true distribution. Remarkably, this means that a semiparametric efficient estimator will be automatically parametrically efficient as well. The aim of this letter is then to derive such a semiparametric, distribution-free estimator, able to achieve the classical CRB for the time-delay and Doppler estimation problem *regardless of the unknown CES noise distribution*.

II. Signal model

A. CES-based signal model with unspecified density

We consider a system which transmits a band-limited signal $s(t)$, with bandwidth B over a carrier frequency f_c ($\lambda_c = c/f_c$, $\omega_c = 2\pi f_c$) from a transmitter T at position $\boldsymbol{p}_T(t)$ to a receiver R at position $\boldsymbol{p}_R(t)$. Assuming a first order approximation, the distance transmitted is $\boldsymbol{p}_{TR} \approx c(\bar{\tau} + \bar{b}t)$, with $\bar{\tau} = \frac{\|\boldsymbol{p}_T(0)-\boldsymbol{p}_R(0)\|}{c}$ and $\bar{b} = \frac{\pm\|v\|}{c}$ with $\boldsymbol{v}$ the relative velocity vector between the transmitter and the receiver. Under the narrowband assumption, the received signal after the baseband demodulation can be expressed as [14], [20], [37]

$$x\left(t;\bar{\boldsymbol{\eta}}\right) = \bar{\alpha} s\left(t-\bar{\tau}\right) e^{-j2\pi f_c\left(\bar{b}(t-\bar{\tau})\right)} + n\left(t\right), \tag{1}$$

with $\bar{\boldsymbol{\eta}} = \left(\bar{\tau}, \bar{b}\right)^T$ and $\bar{\alpha}$ a complex gain. The discrete vector signal model is built from $N = N_1 - N_2 + 1$ samples at $T_s = 1/F_s = 1/B$,

$$\mathbf{x} = \bar{\alpha}\boldsymbol{\mu}(\bar{\boldsymbol{\eta}}) + \mathbf{n}, \tag{2}$$

This work was supported by TéSA.

Lorenzo Ortega is with the DR2I - IPSA and TéSA, 7, Boulevard de la Gare, 31500, Toulouse, France (e-mail: lorenzo.ortega@tesa.prd.fr).

Stefano Fortunati is with SAMOVAR, Télécom SudParis, Institut Polytechnique de Paris, 91120 Palaiseau, France (e-mail: stefano.fortunati@telecom-sudparis.eu)

where each entry of the vectors $\mathbf{x}$, $\boldsymbol{\mu}(\bar{\boldsymbol{\eta}})$ and $\mathbf{n}$ are given by $x_k = x(kT_s)$, $\mu_k(\bar{\boldsymbol{\eta}}) = s(kT_s - \bar{\tau})e^{-j2\pi f_c(\bar{b}(kT_s-\bar{\tau})}$ and $n_k = n(kT_s)$, for $N_1 \leq k \leq N_2$. The i.i.d. noise samples $n(kT_s) \sim CES(0, \bar{\sigma}_n^2, \bar{g})$ are assumed to be CES-distributed with unknown noise power $\bar{\sigma}_n^2$ and unspecified density generator $\bar{g}$ [38]. The unknown deterministic parameters can be gathered in vector $\bar{\boldsymbol{\epsilon}}^\top = (\bar{\sigma}_n^2, \bar{\rho}, \bar{\Phi}, \bar{\boldsymbol{\eta}}^\top) = (\bar{\sigma}_n^2, \bar{\boldsymbol{\theta}}^\top)$, with $\bar{\alpha} = \bar{\rho}e^{j\bar{\Phi}}$ and $\bar{\rho} \in \mathbb{R}^+, 0 \leq \bar{\Phi} \leq 2\pi$. The underling data generating model is then characterized by the following pdf $p_{\bar{\boldsymbol{\epsilon}}}(\mathbf{x}; \bar{\boldsymbol{\epsilon}}) = \Pi_{k=N_1}^{N_2} p_{\bar{\boldsymbol{\epsilon}}}(x_k, \bar{\boldsymbol{\epsilon}})$, with $p_{\bar{\boldsymbol{\epsilon}}}(x_k, \bar{\boldsymbol{\epsilon}}) = CES(\bar{\alpha}\boldsymbol{\mu}(\bar{\boldsymbol{\eta}}), \bar{\sigma}_n^2, \bar{g})$. Note that, since $\bar{g}$ is left unspecified, $p_{\bar{\boldsymbol{\epsilon}}}(\mathbf{x}; \bar{\boldsymbol{\epsilon}})$ cannot be used to derive a MLE for $\bar{\boldsymbol{\epsilon}}$ since the functional form of the likelihood is unknown. For further reference, we recall that, according to the Stochastic Representation Theorem [38, Theo. 3], we have:

$$x_k =_d \bar{\alpha}\mu_k(\bar{\boldsymbol{\eta}}) + \sqrt{Q_k}\bar{\sigma}_n u_k =_d f_k(\bar{\boldsymbol{\theta}}) + \sqrt{\mathcal{Q}}\bar{\sigma}_n u_k, \quad (3)$$

where u_k is a complex uni-variate random variable uniformly distributed on $\mathbb{C}S \triangleq \{u \in \mathbb{C} | |u| = 1\}$, i.e. $u_k \sim U(\mathbb{C}S)$. Moreover $Q_k \triangleq |x_k - f_k(\bar{\boldsymbol{\theta}})|^2/\bar{\sigma}_n^2 =_d \mathcal{Q}$ is a positive random variable, independent from u_k with pdf $p_{\mathcal{Q}}(q) = \delta_g^{-1}\bar{g}(q)$, where $\delta_g \triangleq \int_0^\infty \bar{g}(q)dq$ serves as a normalization constant (see [38, Eq. (19)]). To resolve the known scale ambiguity between $\bar{\sigma}_n$ and $\bar{g}$, we impose that $E\{\mathcal{Q}\} = 1$. This constraint allows us to interpret $\bar{\sigma}_n$ as the *statistical power* P of the data x_k (as discussed in [38, Sec. III.C]).

The underlying semiparametric estimation problem can be cast as follow: *Is it possible to derive a semiparametric efficient estimator of $\bar{\boldsymbol{\theta}}$ in the presence of an unspecified density generator $\bar{g}$?* This question contains two inner aspects:

1) *Definition of an efficiency bound for $\bar{\boldsymbol{\theta}}$.* To this end, we will use the Semiparametric Slepian-Bangs formula derived in [34, eq.(47)].
2) *Derivation of an efficient estimator able to achieve this bound.* We provide an original rank-based (R-) estimator able to satisfy this property.

An important remark is in order here: in the derivation of the bound for $\bar{\boldsymbol{\theta}}$, the noise power $\bar{\sigma}_n^2$ is assumed to be a-priori known. Nevertheless, to overcome this unrealistic requirement, in the derivation of the R-estimator, the true (and unavailable) noise power is substituted with a consistent estimator.

III. Closed-form expression of the SCRB for $\bar{\boldsymbol{\theta}}$

According to the semiparametric theory [39], the lack of a-priori knowledge about the density generator $\bar{g}$ can be taken into account in the derivation of the semiparametric version of the CRB, i.e. the SCRB, on $\bar{\boldsymbol{\theta}}$ by considering $\bar{g}$ as a *functional* nuisance term. Specifically, the SCRB can be defined as the inverse of the so called Semiparametric Efficient Fisher Information Matrix (SFIM). Without any claim of completeness (we refer the reader to [36] for all the details), the SFIM is defined as $\bar{\mathbf{I}}(\bar{\boldsymbol{\theta}}|\bar{g}) \triangleq E\{\bar{\mathbf{s}}_{\bar{\boldsymbol{\theta}}}(x)\bar{\mathbf{s}}_{\bar{\boldsymbol{\theta}}}(x)^H\}$ where the semiparametric efficient score vector $\bar{\mathbf{s}}_{\bar{\boldsymbol{\theta}}}$ is given by:

$$\bar{\mathbf{s}}_{\bar{\boldsymbol{\theta}}}(x) \triangleq \mathbf{s}_{\bar{\boldsymbol{\theta}}}(x) - \Pi(\mathbf{s}_{\bar{\boldsymbol{\theta}}}(x)|\mathcal{T}_{\bar{g}}), \quad (4)$$

where $\mathbf{s}_{\bar{\boldsymbol{\theta}}}(x) \triangleq \nabla_\theta \ln p_{\bar{\boldsymbol{\epsilon}}}(x; \bar{\sigma}^2, \bar{\boldsymbol{\theta}})$ is the score vector evaluated at the true parameter vector $\bar{\boldsymbol{\theta}}$ and $\Pi(\mathbf{s}_{\bar{\boldsymbol{\theta}}}(x)|\mathcal{T}_{\bar{g}})$ is the orthogonal projection of $\mathbf{s}_{\bar{\boldsymbol{\theta}}}$ on the semiparametric nuisance tangent space $\mathcal{T}_{\bar{g}}$ [34], [36] evaluated at the true density generator $\bar{g}$. The explicit calculation of the SFIM for CES-distributed can be obtained through the Semiparametric Slepian-Bangs formula that has been obtained under very general conditions in [34, eq.(47)]. For the case under consideration, we can get a quite surprisingly simplification of this general formula. In fact, since, according to (3), $\bar{\boldsymbol{\theta}}$ parametrizes only the mean of the CES-distributed data, [34, eq. (42)] shows that the projection operator $\Pi(\mathbf{s}_{\bar{\boldsymbol{\theta}}}(x)|\mathcal{T}_{\bar{g}})$ is null for any possible unspecified density generator. In words, this means that *knowing or not knowing the density generator $\bar{g}$ does not have any impact of the asymptotic lower bound for the estimation of $\bar{\boldsymbol{\theta}}$!* As a consequence, the efficient score vector $\bar{\mathbf{s}}_{\bar{\boldsymbol{\theta}}}(x)$ equate the "parametric" score vector $\mathbf{s}_{\bar{\boldsymbol{\theta}}}(x)$ and the SFIM equates the the "parametric" FIM $\mathbf{I}(\bar{\boldsymbol{\theta}}) \triangleq E\{\mathbf{s}_{\bar{\boldsymbol{\theta}}}(x)\mathbf{s}_{\bar{\boldsymbol{\theta}}}^H(x)\}$. Moreover, the score vector $\mathbf{s}_{\bar{\boldsymbol{\theta}}}(x)$ and the related FIM are obtained from [34, eq. (41)] and [34, eq.(47)] respectively as:

$$\mathbf{s}_{\bar{\boldsymbol{\theta}}}(x_k) =_d -2\bar{\sigma}_n^{-1}\sqrt{\mathcal{Q}}\bar{\psi}(\mathcal{Q})\Re\left\{u_k^*\nabla_{\boldsymbol{\theta}} f_k(\bar{\boldsymbol{\theta}})\right\}, \quad (5)$$

$$\mathbf{I}(\bar{\boldsymbol{\theta}}) = \frac{2E\{\mathcal{Q}\bar{\psi}(\mathcal{Q})^2\}}{\bar{\sigma}_n^2}\Re\left\{\left(\frac{\partial\bar{\alpha}\boldsymbol{\mu}(\bar{\boldsymbol{\eta}})}{\partial\boldsymbol{\theta}}\right)^H\left(\frac{\partial\bar{\alpha}\boldsymbol{\mu}(\bar{\boldsymbol{\eta}})}{\partial\boldsymbol{\theta}}\right)\right\}, \quad (6)$$

where $\bar{\psi}(t) \triangleq d\ln\bar{g}(t)/dt$ and $\mathcal{Q}$ is defined in (3) and the expectation is taken with respect to its pdf $p_{\mathcal{Q}}$. Since the SFIM equates the FIM, as direct consequence, we have that

$$\mathrm{SCRB}(\bar{\boldsymbol{\theta}}|\bar{g}) = \mathbf{I}(\bar{\boldsymbol{\theta}})^{-1} = \mathrm{CRB}(\bar{\boldsymbol{\theta}})\ \forall\bar{g}. \quad (7)$$

We conclude this section by noticing that a closed-form expression of $\mathbf{I}(\bar{\boldsymbol{\theta}}|\bar{g})$ can then be obtained as follows [20]:

$$\mathbf{I}(\bar{\boldsymbol{\theta}}) = E\{\mathcal{Q}\bar{\psi}(\mathcal{Q})^2\}\mathbf{K}(\bar{\boldsymbol{\theta}}), \quad \mathbf{K}(\bar{\boldsymbol{\theta}}) \triangleq \frac{2F_s}{\bar{\sigma}_n^2}\Re\left\{\mathbf{Q}\mathbf{W}\mathbf{Q}^H\right\} \quad (8)$$

with $\mathbf{W} = \begin{bmatrix} w_1 & w_2^* & w_3^* \\ w_2 & w_{2,2} & w_4^* \\ w_3 & w_4 & w_{3,3} \end{bmatrix}$ where the matrix elements can be expressed w.r.t. the baseband signal vector $\mathbf{s}$ as: $w_1 = \frac{1}{F_s}\mathbf{s}^H\mathbf{s}$, $w_2 = \frac{1}{F_s^2}\mathbf{s}^H\mathbf{D}\mathbf{s}$, $w_3 = \mathbf{s}^H\boldsymbol{\Lambda}\mathbf{s}$, $w_4 = \frac{1}{F_s}\mathbf{s}^H\mathbf{D}\boldsymbol{\Lambda}\mathbf{s}$, $w_{2,2} = \frac{1}{F_s^3}\mathbf{s}^H\mathbf{D}^2\mathbf{s}$, $w_{3,3} = F_s\mathbf{s}^H\mathbf{V}\mathbf{s}$ and

$$\mathbf{Q} = \begin{bmatrix} 1 & 0 & 0 \\ j\bar{\rho} & 0 & 0 \\ j\bar{\rho}2\pi f_c\bar{b} & 0 & -\bar{\rho} \\ 0 & -j\bar{\rho}2\pi f_c & 0 \end{bmatrix}, \quad (9)$$

$s_k = s(kT_s)$, $N_1 \leq k \leq N_2$ are the entries of $\mathbf{s}$, $\mathbf{D} = \mathrm{diag}\left(\ldots, k, \ldots\right)_{N_1 \leq k \leq N_2}$ and the matrices $\boldsymbol{\Lambda}$ and $\mathbf{V}$ are defined, element by element, as $[\boldsymbol{\Lambda}]_{k,k'} = \left| \begin{array}{l} k' \neq k : \frac{(-1)^{|k-k'|}}{k-k'} \\ k' = k : 0 \end{array} \right.$ and $[\mathbf{V}]_{k,k'} = \left| \begin{array}{l} k' \neq k : \frac{2(-1)^{|k-k'|}}{(k-k')^2} \\ k' = k : \pi^2/3 \end{array} \right.$.

IV. A (semi)parametric efficient R-estimator for $\bar{\boldsymbol{\theta}}$

In the previous section, we showed that, *for the particular signal model given in* (3), the SCRB is equal to the classical CRB. This means that, if we find an estimator that is (asymptotically) efficient w.r.t. the SCRB, it will also be efficient wrt the CRB. In other words, *for the signal model in* (3), **semiparametric efficiency implies classical parametric efficiency**. This means that *the semiparametric estimator, i.e. an estimator that does not rely on the a-priori knowledge of the density generator $\bar{g}$, may achieve the same asymptotic performance of a classical parametric estimator derived under the assumption of a perfect knowledge of $\bar{g}$*. Let us now focus on the derivation of such a semiparametric estimator. We would like to stress that our aim here is not to give an exhaustive review on the derivation of semiparametric estimators. Instead, we will use known results to derive a specific estimator for the signal model in (3). As discussed in [40], a semiparametric efficient estimator can be obtained in two steps: 1) Evaluate a One-Step (OS) estimator under the assumption of known $\bar{g}$. 2) Use rank-based procedures to approximate the previously derived OS estimator when $\bar{g}$ is unknown. It is important to emphasise that the main purpose of the subsections below is to provide all the information needed to implement the R-estimator for the specific application at hand. The reader wishing to understand all the theoretical details can find them in [40]–[42] while an intuitive introduction to R-estimators can be found in the supporting material of [35]. Moreover, in the supporting material of this letter [43], the calculations leading to the explicit form of the R-estimator are clearly detailed.

A. *A One-Step estimator for the signal model* (3)

Introduced by LeCam (see e.g. [44]), an OS estimator is a parametric estimator able to achieve the same asymptotic optimality properties of the Maximum likelihood estimator [36, Sect. 2.3.1]. For the signal model (3), and under the unrealistic assumption of a perfect a-priori knowledge of $\bar{g}$, the OS estimator can be cast as follows:

$$\hat{\boldsymbol{\theta}}_{OS} = \boldsymbol{\theta}^\star + N^{-1/2}[\mathbf{I}(\boldsymbol{\theta}^\star)]^{-1}\Delta_N(\boldsymbol{\theta}^\star) \text{ where} \tag{10}$$

1) $\boldsymbol{\theta}^\star$ is the *preliminary estimator*, i.e. a $\sqrt{N}-$consistent, but not necessarily efficient, estimator of $\bar{\boldsymbol{\theta}}$.
2) $\mathbf{I}(\boldsymbol{\theta}^\star)$ is the FIM in (6) evaluated in $\boldsymbol{\theta}^\star$,
3) $\Delta_N(\boldsymbol{\theta}^\star)$ is the *central sequence* evaluated in $\boldsymbol{\theta}^\star$ s.t.:

$$\Delta_N(\boldsymbol{\theta}) \triangleq N^{-1/2}\sum\nolimits_{k=1}^{N}\mathbf{s}_{\boldsymbol{\theta}}(x_k), \tag{11}$$

where $\mathbf{s}_{\boldsymbol{\theta}}(x_k)$ is given in (5).

Since the terms $\mathbf{I}(\boldsymbol{\theta}^\star)$ and $\Delta_N(\boldsymbol{\theta}^\star)$ depend on the true (but unknown) density generator $\bar{g}$, the OS estimator in (10) cannot be applied in a semiparametric scenario in which $\bar{g}$ is unknown. For this reason, we have to rely on a rank-based, "$\bar{g}$-free" approximation of $\mathbf{I}(\boldsymbol{\theta}^\star)$ and $\Delta_N(\boldsymbol{\theta}^\star)$.

B. *Rank-based, "$\bar{g}$-free" estimator for $\bar{\boldsymbol{\theta}}$*

Ranks owe their importance in robust statistics to the property of being "distribution-free". In particular, rank measurable functions can be used to derive non-parametric estimators, when the data distribution is unknown as in our case. From eqs. (5) and (6), it is immediate to note that $\mathbf{I}(\boldsymbol{\theta}^\star)$ and $\Delta_N(\boldsymbol{\theta}^\star)$ depend on the unavailable $\bar{g}$ through the function $\bar{\psi}$ and the expectation $E\{\mathcal{Q}\bar{\psi}(\mathcal{Q})^2\}$ since it depends on the pdf of $\mathcal{Q}$ that, in turn, depends on $\bar{g}$. Following the general procedure discussed in [35], [40], we can introduce:

$$Q_k^\star \triangleq |x_k - f_k(\boldsymbol{\theta}^\star)|^2/(\sigma_n^\star)^2, \tag{12}$$

$$u_k^\star \triangleq (x_k - f_k(\boldsymbol{\theta}^\star))\,/(\sigma_n^\star\sqrt{Q_k^\star}). \tag{13}$$

We can now introduce the *ranks* $\{r_k\}_{k=N_1}^{N_2}$ of the (continuous) real random variables $\{Q_k^\star\}_{k=N_1}^{N_2}$ in the usual sense mutated form the ordered statistics [45, Ch. 13]. We are now ready to provide a rank-based approximation of $\mathbf{I}(\boldsymbol{\theta}^\star)$ and $\Delta_N(\boldsymbol{\theta}^\star)$. In fact, it can be shown that [45, Ch. 13], [35], for a given "score function" $M(\cdot)$, [1] we have[2]:

$$M\left(\frac{r_k}{N+1}\right) = \sqrt{\mathcal{Q}}\bar{\psi}(\mathcal{Q}) + o_P(1). \tag{14}$$

Consequently, an approximated version of the efficient central sequence $\Delta_N(\boldsymbol{\theta}^\star)$ in (11) can be obtained as:

$$\widetilde{\Delta}_N(\boldsymbol{\theta}^\star) \triangleq \frac{-2}{\sqrt{N}\sigma_n^\star}\sum_{k=N_1}^{N_2} M\left(\frac{r_k}{N+1}\right)\Re\left[(u_k^\star)^*\nabla_{\boldsymbol{\theta}}f_k(\boldsymbol{\theta}^\star)\right]. \tag{15}$$

Moreover, it can be shown that [41], [42] that a consistent estimator of the term $E\{\mathcal{Q}\bar{\psi}(\mathcal{Q})^2\}$ can be obtained as:

$$E\{\mathcal{Q}\bar{\psi}(\mathcal{Q})^2\} = \hat{\alpha}_N + o_P(1), \text{ where} \tag{16}$$

$$\hat{\alpha}_N = \frac{(\sigma_n^\star)^2}{N}\frac{||\widetilde{\Delta}_N(\boldsymbol{\theta}^\star + N^{-1/2}\mathbf{v}^0) - \widetilde{\Delta}_N(\boldsymbol{\theta}^\star)||}{||\mathbf{K}(\boldsymbol{\theta}^\star)\mathbf{v}^0||}, \tag{17}$$

where we introduced the "small perturbation" vector $\mathbf{v}^0 \sim \mathcal{N}(\mathbf{0}, \varrho\mathbf{I})$. As a consequence, an approximation of the FIM in (8) can be obtained as $\widetilde{\mathbf{I}}(\boldsymbol{\theta}^\star) = \hat{\alpha}\mathbf{K}(\boldsymbol{\theta}^\star)$. We are now ready to provide the main result of this letter. In fact, a rank-based, "$\bar{g}$-free" estimator for $\bar{\boldsymbol{\theta}}$, say $\hat{\boldsymbol{\theta}}_R$, can be obtained by substituting in the OS estimator of eq. (10) the approximated versions of $\Delta_N(\boldsymbol{\theta}^\star)$ and $\mathbf{I}(\boldsymbol{\theta}^\star)$, given in eqs. (15) and (17) respectively

$$\hat{\boldsymbol{\theta}}_R = \boldsymbol{\theta}^\star + (\sqrt{N}\hat{\alpha}_N)^{-1}[\mathbf{K}(\boldsymbol{\theta}^\star)]^{-1}\widetilde{\Delta}_N(\boldsymbol{\theta}_n^\star). \tag{18}$$

It is important to stress that $\hat{\boldsymbol{\theta}}_R$ does not depend on the density generator $\bar{g}$ that is left unspecified and depends on the collected data only through the ranks of the real random variables $\{Q_k^\star\}_{k=N_1}^{N_2}$ defined in (12). Remarkably, $\hat{\boldsymbol{\theta}}_R$ satisfies:

$$\sqrt{N}\left(\hat{\boldsymbol{\theta}}_R - \bar{\boldsymbol{\theta}}\right) \xrightarrow{d} \mathcal{N}(\mathbf{0}, [\zeta(M,\bar{g})\mathbf{K}(\bar{\boldsymbol{\theta}})]^{-1}),\ \forall\bar{g},$$

where $\zeta(M,\bar{g})$ is defined in [42]. We underline that the estimator in (17) is consistent but not efficient. Therefore, a more efficient estimator of $\hat{\alpha}$ can improve the asymptotic performance of the proposed R-estimator in (18).

[1] The family of score functions is defined in [46, Sect. 2.2], [45, Ch. 13]

[2] We write: $x_l = o_P(1)$ if $\lim_{l\to\infty}\Pr\{|x_l| \geq \epsilon\} = 0, \forall\epsilon > 0$ (*convergence in probability to 0*)

C. Choice of $\boldsymbol{\theta}^\star$, $(\sigma_n^\star)^2$, and $M(\cdot)$

As previously stated, the preliminary estimator $(\boldsymbol{\theta}^\star)^\top = [\rho^\star, \Phi^\star, (\boldsymbol{\eta}^\star)^\top]$ needs to be a $\sqrt{N}$-consistent (but not necessarily efficient) estimator of $\bar{\boldsymbol{\theta}}$. So, a perfect candidate would be the MMLE derived in [24], [27]. Similarly, as preliminary estimator of the noise variance, we can use:

$$(\sigma_n^\star)^2 = \|\boldsymbol{x} - \rho^\star e^{j\Phi^\star}\boldsymbol{\mu}(\boldsymbol{\eta}^\star)\|^2/N. \tag{19}$$

Regarding the "score function" $M(\cdot)$ many choices are possible (see e.g. [35]). However, the one that provide a good trade of between semiparametric efficiency and robustness is the complex *van der Waerden* score function $M_{vdW}(t) \triangleq \sqrt{\Phi_G^{-1}(t)}, t \in (0,1)$ where Φ_G^{-1} indicates the inverse function of the cdf of a Gamma-distributed random variable with parameters $(1,1)$.

D. Explicit calculation of $\mathbf{K}(\boldsymbol{\theta}^\star)$ and $\widetilde{\Delta}_N(\boldsymbol{\theta}^\star)$

$\mathbf{K}(\boldsymbol{\theta}^\star)$ can be simply calculated directly by substituting the estimates of the MMLE $\boldsymbol{\theta}^\star$ in equation (8). To calculate $\widetilde{\Delta}_N(\boldsymbol{\theta}^\star)$, we first derive:

$$\nabla_{\boldsymbol{\theta}} f_k(\boldsymbol{\theta}^\star) = \mathbf{Q}^\star \boldsymbol{\vartheta}(kT_s;\boldsymbol{\theta}^\star) e^{j\Phi^\star} e^{-j\omega_c b^\star (kT_s - \tau^\star)}, \tag{20}$$

with $\mathbf{Q}^\star$ computed by substituting the estimates of the MMLE in (9) and $\boldsymbol{\vartheta}(kT_s;\boldsymbol{\theta}^\star) = \begin{bmatrix} s(kT_s-\tau^\star) \\ (t-\tau^\star)s(kT_s-\tau^\star) \\ s^{(1)}(kT_s-\tau^\star)\end{bmatrix}$. Let us define the entries of the vector $\boldsymbol{\mathcal{U}}^\star$ as $\mathcal{U}_k^\star = M\left(\frac{r_k}{N+1}\right)(u_k^\star)^*$. Then, assuming a band-limited signal, Shannon-Nyquist's theorem can be used to compute $\widetilde{\Delta}_N(\boldsymbol{\theta}^\star)$ since

$$\lim_{(N_1,N_2)\to(-\infty,\infty)} T_s \left(\sum_{k=N_1}^{N_2} \mathcal{U}_k^* \boldsymbol{\vartheta}(kT_s;\boldsymbol{\theta}^\star) e^{-j\omega_c b^\star (kT_s-\tau^\star)}\right) =$$
$$\int_{-\infty}^{\infty} \mathcal{U}^*(t)\boldsymbol{\vartheta}(t;\boldsymbol{\theta}^\star) e^{-j\omega_c b^\star (t-\tau^\star)} dt = (w_{e_1}, w_{e_2}, w_{e_3})^\top.$$

where the integral solutions can be found in [47],

$$w_{e_1} = \frac{1}{F_s}(\boldsymbol{\mathcal{U}}^\star)^H \boldsymbol{V}^{\Delta,0}\left(\frac{\tau^\star}{T_s}\right)\boldsymbol{U}\left(\frac{f_c b^\star}{F_s}\right)\boldsymbol{s},$$
$$w_{e_2} = \frac{1}{F_s^2}(\boldsymbol{\mathcal{U}}^\star)^H \boldsymbol{V}^{\Delta,0}\left(\frac{\tau^\star}{T_s}\right)\boldsymbol{U}\left(\frac{f_c b^\star}{F_s}\right)\boldsymbol{D}\mathbf{s},$$
$$w_{e_3} = (\boldsymbol{\mathcal{U}}^\star)^H \boldsymbol{V}^{\Delta,1}\left(\frac{\tau^\star}{T_s}\right)\boldsymbol{U}\left(\frac{f_c b^\star}{F_s}\right)\mathbf{s} \;+ jw_c b^\star w_{e_1},$$

with $\mathbf{U}(p) = \mathrm{diag}(\cdots, e^{-j2\pi pk}, \cdots)_{N_1\le k\le N_2}$, $[\mathbf{V}^{\Delta,1}(q)]_{k,l} = \frac{1}{k-l-q}(\cos(\pi(k-l-q)) - \mathrm{sinc}(k-l-q))$ and $[\mathbf{V}^{\Delta,0}(q)]_{k,l} = \mathrm{sinc}(k-l-q)$. Then,

$$\widetilde{\Delta}_N(\boldsymbol{\theta}^\star) = (2F_s)/(\sqrt{N}\sigma^\star)\Re\left\{e^{j\Phi^\star}\mathbf{Q}^\star \boldsymbol{w}_e\right\}. \tag{21}$$

V. Simulation and Discussion

We consider a scenario where a GPS L1 C/A signal [10] is received by a GNSS receiver. We set a true signal model where the noise is sampled from a complex centered t-distribution [32, Sec. 4.6.1.1] with $\upsilon = 1.5$ degrees of freedom (or *shape parameter*) that control the level of non-Gaussianity and a scale parameter μ. The second-order modular variate $\mathcal{Q}$ of a

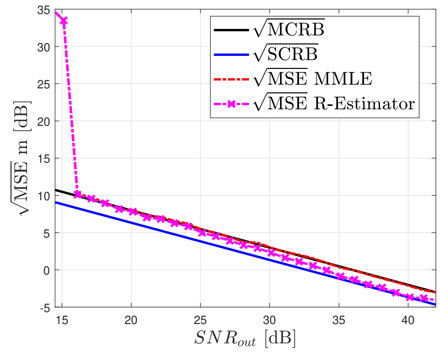

Fig. 1. RMSE of the MMLE and R-estimator in (18) of the time-delay.

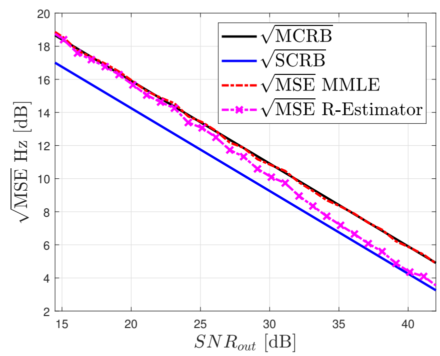

Fig. 2. RMSE of the MMLE and R-estimator in (18) of the Doppler.

t-distribution is an F-distributed random variable with parameter 2 and 2υ, i.e., $\mathcal{Q} \sim \mu^{-1}F_{2,2\upsilon}$ [38, Sec. IV.A]. Then, to meet the constraint $E\{\mathcal{Q}\} = 1$, the scale must be set as $\mu = \frac{\upsilon}{(\upsilon-1)}$. Moreover, for the t-distribution $E\{\mathcal{Q}\bar{\psi}(\mathcal{Q})^2\} = \frac{\mu(\upsilon+1)}{(\upsilon+2)}$ as it has been shown in [34] and $SNR_{out} = |\bar{\alpha}|^2\mathbf{s}^H\mathbf{s}/\bar{\sigma}_n^2$. Figures 1 and 2 present the root mean square error (RMSE) results of the MMLE and the R-estimator derived in (18) for the parameters of interest $\bar{\boldsymbol{\eta}}^T = [\bar{\tau}, \bar{b}]$, as a function of the SNR_{out}. This analysis considers a setup with a GNSS receiver operating at a sampling frequency $F_s = 4$MHz and an integration time of 1 ms. The results are based on 1000 Monte Carlo iterations. The results show that the RMSE ($\sqrt{MSE}$) of the MMLE asymptotically converge to the MCRB, which matches with the Complex-Gaussian CRB. This results was already shown in [30]. Moreover, we show the RMSE of the estimator derived in (18) asymptotically converge to the SCRB/CRB in (6). The reader can find additional numerical results on the estimator's performance with other parameters and distributions in [43].

VI. Conclusion

The purpose of this letter was to introduce new insights into the theory of time-delay and Doppler estimation. In particular, we derived an original rank-based, distribution free, R-estimator which is asymptotically parametrically efficient regardless the unspecified noise CES distribution. To validate the efficiency of the derived R-estimator, we compared its MSE with the relevant semiparametric bound, which is proven to equate (in the case of time-delay and Doppler estimation) the classical CRB evaluated for the true noise distribution.

REFERENCES

[1] D. A. Swick, "A Review of Wideband Ambiguity Functions," Tech. Rep. 6994, Naval Res. Lab., Washington DC, 1969.

[2] H. L. Van Trees, *Detection, Estimation, and Modulation Theory, Part III: Radar – Sonar Signal Processing and Gaussian Signals in Noise*, J. Wiley & Sons, 2001.

[3] U. Mengali and A. N. D'Andrea, *Synchronization Techniques for Digital Receivers*, Plenum Press, New York, USA, 1997.

[4] H. L. Van Trees, *Optimum Array Processing*, Wiley-Interscience, New-York, 2002.

[5] D. W. Ricker, *Echo Signal Processing*, Kluwer Academic, Springer, New York, USA, 2003.

[6] J. Chen, Y. Huang, and J. Benesty, "Time delay estimation," in *Audio Signal Processing for Next-Generation Multimedia Communication Systems*, Y. Huang and J. Benesty, Eds., chapter 8, pp. 197–227. Springer, Boston, MA, USA, 2004.

[7] B. C. Levy, *Principles of Signal Detection and Parameter Estimation*, Springer, 2008.

[8] F. Bouchereau D. Munoz, C. Vargas, and R. Enriquez, *Position Location Techniques and Applications*, Academic Press, Oxford, GB, 2009.

[9] J. Yan et al., "Review of range-based positioning algorithms," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 28, no. 8, pp. 2–27, Aug. 2013.

[10] P. J. G. Teunissen and O. Montenbruck, Eds., *Handbook of Global Navigation Satellite Systems*, Springer, Switzerland, 2017.

[11] S. M. Kay, *Fundamentals of Statistical Signal Processing: Estimation Theory*, Prentice-Hall, Englewood Cliffs, New Jersey, USA, 1993.

[12] H. L. Van Trees and K. L. Bell, Eds., *Bayesian Bounds for Parameter Estimation and Nonlinear Filtering/Tracking*, Wiley/IEEE Press, NY, USA, 2007.

[13] P. Stoica and A. Nehorai, "Performances study of conditional and unconditional direction of arrival estimation," *IEEE Trans. Acoust., Speech, Signal Process.*, vol. 38, no. 10, pp. 1783–1795, Oct. 1990.

[14] A. Dogandzic and A. Nehorai, "Cramér-Rao bounds for estimating range, velocity, and direction with an active array," *IEEE Trans. Signal Process.*, vol. 49, no. 6, pp. 1122–1137, June 2001.

[15] N. Noels, H. Wymeersch, H. Steendam, and M. Moeneclaey, "True Cramér-Rao bound for timing recovery from a bandlimited linearly modulated waveform with unknown carrier phase and frequency," *IEEE Trans. on Communications*, vol. 52, no. 3, pp. 473–483, March 2004.

[16] T. Zhao and T. Huang, "Cramér-Rao lower bounds for the joint delay-doppler estimation of an extended target," *IEEE Trans. Signal Process.*, vol. 64, no. 6, pp. 1562–1573, March 2016.

[17] Y. Chen and R. S. Blum, "On the Impact of Unknown Signals on Delay, Doppler, Amplitude, and Phase Parameter Estimation," *IEEE Trans. Signal Process.*, vol. 67, no. 2, pp. 431–443, Jan. 2019.

[18] Q. Jin, K. M. Wong, and Z.-Q. Luo, "The Estimation of Time Delay and Doppler Stretch of Wideband Signals," *IEEE Trans. Signal Process.*, vol. 43, no. 4, pp. 904–916, April 1995.

[19] X.X. Niu, P. C. Ching, and Y.T. Chan, "Wavelet Based Approach for Joint Time Delay and Doppler Stretch Measurements," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 35, no. 3, pp. 1111–1119, July 1999.

[20] D. Medina, L. Ortega, J. Vilà-Valls, P. Closas, François Vincent, and E. Chaumette, "Compact CRB for delay, doppler and phase estimation - application to GNSS SPP & RTK performance characterization," *IET Radar, Sonar & Navigation*, vol. 14, no. 10, pp. 1537–1549, Sep. 2020.

[21] P. Das, J. Vilà-Valls, F. Vincent, L. Davain, and E. Chaumette, "A New Compact Delay, Doppler Stretch and Phase Estimation CRB with a Band-Limited Signal for GenE. Remote Sensing Applications," *Remote Sensing*, vol. 12, no. 18, pp. 2913, Sep. 2020.

[22] C. Lubeigt, L. Ortega, J. Vilà-Valls, L. Lestarquit, and E. Chaumette, "Joint delay-doppler estimation performance in a dual source context," *Remote Sensing*, vol. 12, no. 23, 2020.

[23] H. McPhee, L. Ortega, J. Vilà-Valls, and E. Chaumette, "Accounting for acceleration—signal parameters estimation performance limits in high dynamics applications," *IEEE Transactions on Aerospace and Electronic Systems*, vol. 59, no. 1, pp. 610–622, 2023.

[24] C. Lubeigt, L. Ortega, J. Vilà-Valls, and E. Chaumette, "Untangling first and second order statistics contributions in multipath scenarios," *Signal Processing*, vol. 205, pp. 108868, 2023.

[25] H. McPhee, L. Ortega, J. Vilà-Valls, and E. Chaumette, "On the accuracy limits of misspecified delay-doppler estimation," *Signal Processing*, vol. 205, pp. 108872, 2023.

[26] L. Ortega, J. Vilà-Valls, and E. Chaumette, "Theoretical evaluation of the GNSS synchronization performance degradation under interferences," in *Proc. of ION GNSS*, Denver, CO, USA, Sep. 2022.

[27] L. Ortega, C. Lubeigt, J. Vilà-Valls, and E. Chaumette, "On gnss synchronization performance degradation under interference scenarios: Bias and misspecified cramér-rao bounds," *NAVIGATION: Journal of the Institute of Navigation*, vol. 70, no. 4, 2023.

[28] L. Ortega and S. Fortunati, "Misspecified Time-delay and Doppler estimation over high dynamics non-Gaussian scenarios," in *2024 32nd European Signal Processing Conference (EUSIPCO)*, Lyon, France, August 2024, pp. 2297–2301.

[29] L. Ortega and S. Fortunati, "Misspecified time-delay and doppler estimation over non gaussian scenarios," in *ICASSP 2024-2024 IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP)*. IEEE, 2024, pp. 9346–9350.

[30] S. Fortunati and L. Ortega, "On the efficiency of misspecified gaussian inference in nonlinear regression: application to time-delay and doppler estimation," *Signal Processing*, vol. 225, pp. 109614, 2024.

[31] S. Fortunati, F. Gini, M. S Greco, and C. D Richmond, "Performance bounds for parameter estimation under misspecified models: Fundamental findings and applications," *IEEE Signal Process. Mag.*, vol. 34, no. 6, pp. 142–157, 2017.

[32] S. Fortunati, F. Gini, and M. S. Greco, "Chapter 4 - Parameter bounds under misspecified models for adaptive radar detection," in *Academic Press Library in Signal Processing, Volume 7*, Rama Chellappa and Sergios Theodoridis, Eds., pp. 197–252. Academic Press, 2018.

[33] S. Fortunati, F. Gini, M.S. Greco, A. M. Zoubir, and M. Rangaswamy, "Semiparametric inference and lower bounds for real elliptically symmetric distributions," *IEEE Transactions on Signal Processing*, vol. 67, no. 1, pp. 164–177, 2019.

[34] S. Fortunati, F. Gini, M.A. Sabrina Greco, A.M. Zoubir, and M. Rangaswamy, "Semiparametric CRB and Slepian-Bangs formulas for complex elliptically symmetric distributions," *IEEE Transactions on Signal Processing*, vol. 67, no. 20, pp. 5352–5364, 2019.

[35] S. Fortunati, A. Renaux, and F. Pascal, "Robust semiparametric efficient estimators in complex elliptically symmetric distributions," *IEEE Transactions on Signal Processing*, vol. 68, pp. 5003–5015, 2020.

[36] S. Fortunati, "Semiparametric estimation in elliptical distributions," in *Elliptically Symmetric Distributions in Signal Processing and Machine Learning*, J.P. Delams, M.N. El Korso, S. Fortunati, and F. Pascal, Eds. 2024, Springer.

[37] M. I. Skolnik, *Radar Handbook*, McGraw-Hill, New York, USA, 3rd edition, 1990.

[38] E. Ollila, D. E. Tyler, V. Koivunen, and H. V. Poor, "Complex elliptically symmetric distributions: Survey, new results and applications," *IEEE Transactions on Signal Processing*, vol. 60, no. 11, pp. 5597–5625, 2012.

[39] P.J. Bickel, C.A.J Klaassen, Y. Ritov, and J.A. Wellner, *Efficient and Adaptive Estimation for Semiparametric Models*, Johns Hopkins University Press, 1993.

[40] M. Hallin and Bas J. M. Werker, "Semi-parametric efficiency, distribution-freeness and invariance," *Bernoulli*, vol. 9, no. 1, pp. 137–165, 2003.

[41] M. Hallin and D. Paindaveine, "Semiparametrically efficient rank-based inference for shape I. Optimal rank-based tests for sphericity," *The Annals of Statistics*, vol. 34, no. 6, pp. 2707–2756, 2006.

[42] Marc Hallin, Hannu Oja, and Davy Paindaveine, "Semiparametrically efficient rank-based inference for shape II. Optimal *R*-estimation of shape," *The Annals of Statistics*, vol. 34, no. 6, pp. 2757–2789, 2006.

[43] L. Ortega and S. Fortunati, "Supporting materials: Robust semiparametric efficient estimator for time delay and Doppler estimation," Tech. Rep., 2025.

[44] Lucien Le Cam and Grace Lo Yang, *Locally Asymptotically Normal Families*, pp. 52–98, Springer US, New York, NY, 1990.

[45] A. W. van der Vaart, *Asymptotic Statistics*, Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, 1998.

[46] M. Hallin, D. Paindaveine, and T. Verdebout, "Optimal rank-based testing for principal components," *The Annals of Statistics*, vol. 38, no. 6, pp. 3245–3299, 12 2010.

[47] C. Lubeigt, L. Ortega, J. Vilà-Valls, and E. Chaumette, "Developments for MCRB Computation in Multipath Scenarios," 2022.