

# Random Propagation Times for Ultrasonics through Polyethylene

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## Abstract

Low power ultrasonics are used for testing high density polyethylene pipe material. Attenuation and velocity give valuable information on the material in situ and without damages. In this paper we revisit recent data in the frequency band (4,10) megahertz. We prove that propagation is equivalent to random delays following stable probability laws. Moreover, the emergence of a companion noise non-detectable by devices is compliant with the law of conservation of energy.

*Keywords*: polyethylene, ultrasonics, linear filtering, stable probability law, random propagation times.

## 1 Introduction

Effects of ultrasonics on polyethylene have been studied for a long time [1], [2], [3], [4]. Utilization of high density pipe material in nuclear industry has renewed the interest for this topic [5], [6]. In particular, paper [5] provides interesting results about attenuation  $\alpha(f)$  and phase velocity  $v(f)$  of ultrasounds in the frequency band (4, 10) megahertz.

Investigations derive from the time causal theory deduced from Kramers-Krönig relations [1], [7], [8]. We are in the case where attenuation follows a power law, i.e

$$\alpha(f) = \alpha_1 f^y. \quad (1)$$

$f$  is the frequency in MHz,  $v(f)$  in  $\text{m} \cdot \mu\text{s}^{-1}$  and  $\alpha(f)$  in  $\text{m}^{-1}$ . Parameters verify the conditions  $\alpha_1 > 0, 0 < y < 2$ .  $\alpha_1$  depends on the system of

units but not  $y$ . The velocity  $v(f)$  is linked to  $\alpha(f)$  through the approximate relation

$$v(f) \approx v_0 + \left(\frac{v_0}{\pi}\right)^2 \frac{\alpha_1}{y-1} f^{y-1} \quad (2)$$

when  $y > 1$ , with (theoretically)  $v_0 = v(0)$  [1]. In [5], a sample is studied at three different temperatures ( $42^\circ 5, 20^\circ$  and  $8^\circ 1$ ). Attenuation and velocity of monochromatic waves in the frequency band (4, 10) MHz are reported and the paper focuses mainly on uncertainties and regression intervals of measurements. Two figures summarize measurements, the first one shows attenuations and the second one provides velocities. The first one is accurate and relation (1) is well verified. Figure 1 here transforms data coming from figure 4 in [5] in logarithmic coordinates, which leads to a set of three lines of equation ( $f$  in MHz,  $\alpha$  in  $\text{m}^{-1}$ )

$$\begin{array}{ll} 42^\circ 5 & \ln \alpha = 1.16 \ln f + 3.29 \\ 20^\circ & \ln \alpha = 1.185 \ln f + 3.02 \\ 8^\circ 1 & \ln \alpha = 1.262 \ln f + 2.81 \end{array} \quad (3)$$

and equivalently

$$\begin{array}{lll} & y & \alpha_1 \\ 42^\circ 5 & 1.16 & 26.9 \\ 20^\circ & 1.185 & 20.5 \\ 8^\circ 1 & 1.262 & 16.6 \end{array} \quad (4)$$

We retrieve similar values in other papers [4], [6] ( $y$  around or equal to 1). Data on velocity are more difficult to translate. Paper [5] groups the three curves of interest in figure 5. Because they are very well separated, their usable set of values is very small with respect to the full scale (some 10

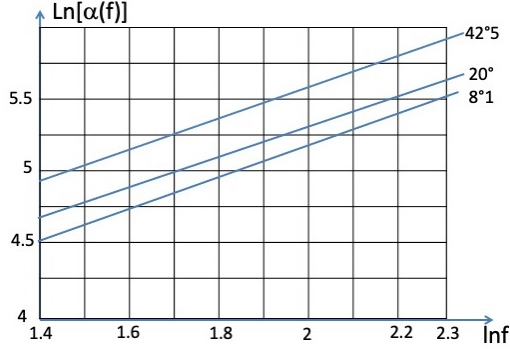


Figure 1: Attenuation  $\alpha(f)$  as a power function.

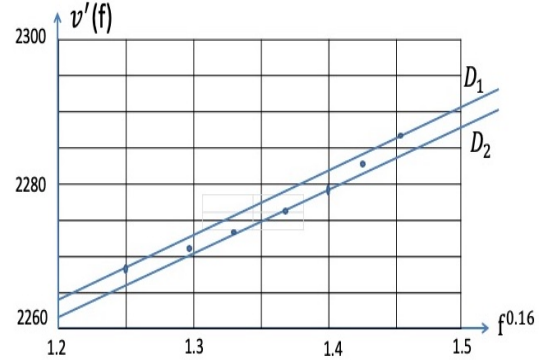


Figure 2:  $v'(f) = 3f + 2256$  as function of  $f^{0.16}$ .

to 20 units with respect to 450). Each curve can be viewed as a line (in the frequency set of interest). A thin curvature can be detected but not quantified because of uncertainties in measurements. We can approximate velocities by lines of equation ( $v(f)$  is in  $m \cdot \mu s^{-1}$ )

$$\begin{array}{ll} 42^\circ & 10^6 v(f) = 3f + 2256 \\ 20^\circ & 10^6 v(f) = 2.6f + 2419 \\ 8^\circ & 10^6 v(f) = 2.3f + 2509 \end{array} \quad (5)$$

Szabo predicted the shape (2) for  $v(f)$  [1], [9], [10]. When we choose the linearity for  $v(f)$  with respect to  $f$  in (5), we will not have the same property with respect to  $f^{y-1}$ . Nevertheless, the gap is not very large. Figure 2 shows points of the curve ( $f^{0.16}, 3f + 2256$ ). The curvature is no longer viewable in the scale chosen in [5] (the gap between both lines on figure 2 is equal to 2.5, compared with 2600-2150=450 in figure 5 of [5]). Any line between  $D_1$  and  $D_2$  is a good approximation of  $v(f)$ . Consequently,  $v(f)$  can be viewed as linearly linked to  $f$  as well as  $f^{y-1}$  in the frequency band (6, 10) MHz (and also to  $f^{1-y}$ , see below).

In the section below, we show that data in [5] can be explained in the frame of random propagation times with stable probability laws. We discuss the causality problem in section 3 and appendices give supplementary material on random propagation times.

## 2 Random propagation times and stable probability laws

### 2.1 Random propagation times

We assume that the random process  $\mathbf{A} = \{A(t), t \in \mathbb{R}\}$  is stationary with characteristic functions (in the probability sense) [11]

$$\psi(f) = \mathbb{E} \left[ e^{-2i\pi f A(t)} \right], \phi(f, \tau) = \mathbb{E} \left[ e^{-2i\pi f (A(t) - A(t-\tau))} \right]$$

which do not depend on  $t$  ( $\mathbb{E}[\dots]$  is for the mathematical expectation or ensemble mean). The process  $\mathbf{U} = \{U(t), t \in \mathbb{R}\}$  is defined as

$$U(t) = e^{2i\pi f_0 (t - A(t))} \quad (6)$$

for some  $f_0$ .  $\mathbf{U}$  results from the monochromatic wave  $e^{2i\pi f_0 t}$  received at some place after some trajectory of random duration  $A(t)$ . We have

$$U(t) = \psi(f_0) e^{2i\pi f_0 t} + V(t) \quad (7)$$

where the zero-mean process  $\mathbf{V}$  is stationary with [12]:

$$\mathbb{E} [V(t) V^*(t - \tau)] = e^{2i\pi f_0 \tau} \left[ \phi(f_0, \tau) - |\psi(f_0)|^2 \right]. \quad (8)$$

If this quantity converges to 0 with  $1/\tau$  quickly enough,  $\mathbf{V}$  will be a "noise" with continuous spectrum. In the case of sonics or ultrasonics,  $A(t)$  models the molecular agitation which induces spectra beyond ultrasonics. Devices which treat  $\mathbf{U}$  are matched to neighbors of  $f_0$  and then will not be

affected by the noise  $\mathbf{V}$  (see [13], [14], [15] and appendices).

We do not accurately know properties of the noise  $\mathbf{V}$ , and this part of  $\mathbf{U}$  is not measured. Nevertheless, this component meets a fundamental property:  $e^{2i\pi f_0 t}$  (the transmitted process) and  $\mathbf{U}$  (the received process) have the same power. The power measured by devices is  $|\psi(f_0)|^2$ , and the power of losses is

$$\mathbb{E} \left[ |V(t)|^2 \right] = 1 - |\psi(f_0)|^2.$$

Fast variations of  $A(t)$  (the changes in propagation times) are sufficient to explain arbitrary losses (see appendices).

## 2.2 Stable probability laws

The (real) random variable  $A$  follows a stable probability law when its characteristic function verifies [16], [17], [18]

$$\begin{aligned} \psi(f) &= \mathbb{E} \left[ e^{-2i\pi f A} \right] \\ &= \exp \left[ -im(2\pi f) - c(2\pi f)^b (1 + i\beta\theta(f)) \right] \end{aligned} \quad (9)$$

where  $f > 0$ ,  $m \in \mathbb{R}$ ,  $c > 0$ ,  $0 < b \leq 2$ ,  $-1 \leq \beta \leq 1$ ,  $\psi(f) = \psi^*(-f)$  and

$$\theta(f) = \tan \frac{\pi b}{2}, b \neq 1 \text{ and } \theta(f) = \frac{2}{\pi} \ln(2\pi f), b = 1.$$

Roughly, r.v  $A_n$ ,  $n \in \mathbb{N}$  are stable when linear combinations are stable (with the same  $b$ ). It is a generalization of Gaussian laws and central limit theorems. Physicists are reluctant to use them because only three stable laws have a reduced shape, the Gaussian ( $b = 2$ ), Cauchy ( $b = 1, \beta = 0$ ) and Levy ( $b = 1/2, \beta = \pm 1$ ) laws.

## 2.3 Application to ultrasonics

We return to the data in [5]. The propagation on the unit length (1m) is a linear time-invariant (LTI) filter  $\mathcal{H}$  of complex gain  $H(f)$  defined by

$$H(f) = \exp \left[ -\alpha_1 f^y - i \frac{2\pi f}{v(f)} \right]. \quad (10)$$

The output of the filter  $H(f_0) e^{2i\pi f_0 t}$  corresponds to the input  $e^{2i\pi f_0 t}$ .

The purpose of this paper is to prove that random propagation times with stable probability laws

are able to explain  $H(f)$ . Equivalently, the output of the filter  $\mathcal{H}$  can be written as (6), knowing that the term  $\mathbf{V}$  in (7) is neglected. From (7), (9) and disregarding the "noise"  $\mathbf{V}$ , we have (using (9) and (10) with  $y > 1$ )

$$\begin{aligned} b &= y \\ c(2\pi)^b &= \alpha_1 \\ m + c(2\pi f)^{b-1} \beta \tan \frac{\pi b}{2} &= \frac{2\pi f}{v(f)} \end{aligned} \quad (11)$$

Furthermore, we have seen that curves  $(f^{y-1}, v(f))$  are close to lines, in accordance with the Szabo's theory. Figures 3 to 5 show that the same property is true for  $(f^{y-1}, v^{-1}(f))$ . It is not a paradox, considering that, in (2), the term  $v_0$  is large with respect to  $\left(\frac{v_0}{\pi}\right)^2 \frac{\alpha_1}{y-1}$ . So, the following equivalence is legitime

$$v^{-1}(f) \simeq \frac{1}{v_0} - \frac{\alpha_1}{\pi^2 (y-1)} f^{y-1}. \quad (12)$$

In figures 3, 4 and 5 parallel lines  $D_1$  and  $D_2$  bound good interpolations for the curve  $(f^{y-1}, v^{-1}(f))$  which is represented by points on the frequency interval (4, 10). They are deduced from data through (5).  $D_1$  and  $D_2$  define a region for a good linear interpolation of  $v^{-1}(f)$  with respect to  $f^{y-1}$ . We have to determine  $m$  and  $\beta$ . It suffices to calculate  $\beta$  so that the slope of the line (in parametric coordinates)

$$\Delta_m : \left( f^{y-1}, m + c(2\pi f)^{y-1} \beta \tan \frac{\pi y}{2} \right)$$

is the same as the slope of  $D_1$  and  $D_2$ .  $m$  is determined when  $\Delta_m$  is between  $D_1$  and  $D_2$ . Uncertainty is measured by the distance between both lines (below the meter). Figures 3 to 5 illustrate the method. In all cases  $\beta$  is close to 1 (see the following section for links with causality).

Finally, we obtain the parameters of equivalent stable probability laws

	$b = y$	$c$	$m$	$\theta$	$\beta$	
$42^\circ 5$	1.16	3.19	465	-3.9	0.95	(13)
$20^\circ$	1.185	2.32	425	-3.34	0.98	
$8^\circ 1$	1.262	1.63	397	-2.29	0.91	

Reciprocally, we deduce the missing parameter  $v_0$  of Szabo's formula from (12) :

$$v_0 = \frac{1}{m} = \begin{cases} 2150 & (42^\circ) \\ 2353 & (20^\circ) \\ 2519 & (8^\circ 1) \end{cases}$$

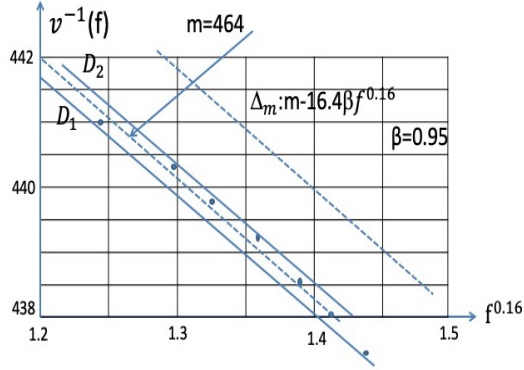


Figure 3:  $\nu^{-1}(f) = (3f + 2256)^{-1}$  as function of  $f^{0.16}$ .

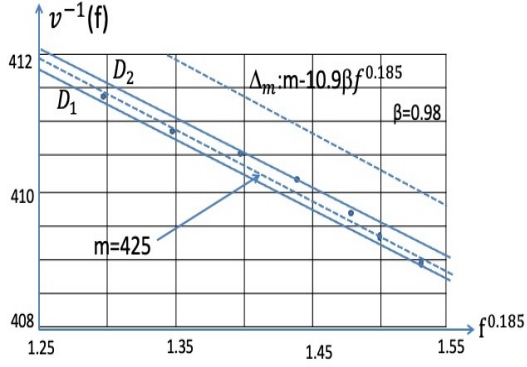


Figure 4:  $\nu^{-1}(f) = (2.6f + 2419)^{-1}$  as function of  $f^{0.185}$ .

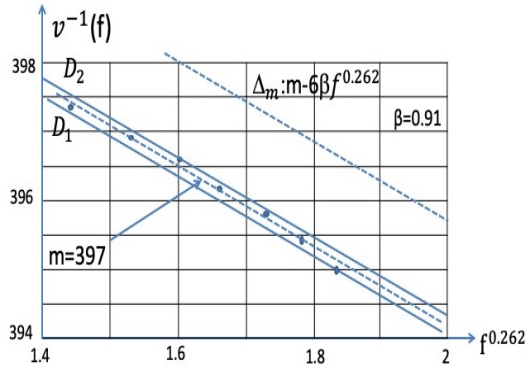


Figure 5:  $\nu^{-1}(f) = (2.3f + 2509)^{-1}$  as function of  $f^{0.262}$ .

in  $\text{m.s}^{-1}$  (in (13)  $m$  is in  $\mu\text{s.m}^{-1}$ ). Finally, random propagation times with stable probability laws are able to model effects of ultrasonics through polyethylene studied in [5], if the "noise" part defined in section 2.2 is disregarded (see appendices 1 and 2).

### 3 Causality

Causality derives from the fact that "an effect does not precede a cause". In physics, this property leads to Kramers-Krönig relations which link attenuation and velocity of waves. They are at the basis of Szabo's theory, though it is proved that causality is not available for attenuations as (1) for  $y \geq 1$  but approached [19].

Causality is explained through notions of complex gains and impulse responses of linear time-invariant filters (LTI) which model the mechanism of some transformation. Computations are achieved from convolution products or Fourier transforms. Actually, the notion of LTI is based on the linearity and only on the additional property:

**a LTI transforms inputs  $e^{2i\pi ft}$   
in outputs  $H(f)e^{2i\pi ft}$ .**

The complex gain is  $H(f)$ , its modulus is the "attenuation" and the argument provides the "velocity" of the monochromatic wave (at the frequency  $f$ ). The property in **bold** is an alternative definition of a LTI, and is never discussed.

The "attenuation" contradicts the first principle of thermodynamics, and a good model has to explain losses. When outputs show a non monochromatic term, it is generally attributed to internal noise of the measuring apparatus or to external sources and not at all to the medium. When we have to solve some physical equation, we first look for monochromatic solutions and generalized by linearity. However, it is not proved that the random character of the medium does not hold changes of the shape of the wave  $e^{2i\pi ft}$ . About "velocity", small gaps from a mean value are difficult to measure, but small values may have big consequences. Admitting the existence of a value which does not vary and independent from microscopic states of the medium, is a gamble.

Random propagation times  $\mathbf{A}$  explain both attenuations and velocities. Unfortunately, explanations are limited by the lack of knowledge of internal links of  $\mathbf{A}$  (the functions  $\phi(f, \tau)$ ). The process  $\mathbf{V}$  contains losses.  $\mathbf{V}$  is stationary, it derives from linearity, but not from a LTI (a continuous power spectrum cannot come from a monochromatic wave). In other situations, for instance in electromagnetic propagation, monochromatic waves are broadened. The part  $\mathbf{V}$  appears and it is the preponderant part, and the propagation cannot be modelled only by LTI [20], [21].

In the random propagation framework, the propagation results in a sum of two processes. The first one is a LTI which explains measurements and the second one explains losses, but qualitatively (it is not viewed by devices). The complex gain of the LTI is the characteristic function  $\psi(f)$  of a stable probability law and the impulse response is the related probability density  $C(t)$  :

$$C(t) = \int_{-\infty}^{\infty} \psi(f) e^{2i\pi ft} df$$

Causality is:  $C(t) = 0$  for  $t < 0$ . The property is verified only when [16], [17], [18], [19]

$$b < 1, \beta = \pm 1.$$

In [5], we have  $b = y > 1$ . The causality property is not met. Nevertheless, the parameter  $\beta$  rules the asymmetry of the probability function  $C(t)$ . The larger is  $|\beta|$  and the more asymmetric is  $C(t)$ . The values  $\beta = 1$  (for  $b \neq 1$ ) and  $\beta = -1$  (for  $b = 1$ ) minimize the weight of probability in  $\mathbb{R}^-$ . They are the best values for a near causality. Values of  $\beta$  deduced from data in [5] and given in (13) are close to 1 and they provide a strong argument in favor of the model.

Figure 6 illustrates this assertion when  $b = y = 1.16$  (other cases are very similar). They show  $C(t)$  for values in (13), and for different  $\beta$ . The value  $\beta = 1$  is the most favourable, and, for  $\beta > 1$ ,  $C(t)$  takes negative values, and then is no longer a probability law. The value of  $m$  ( $m \approx 465\mu s$ ) is very large in front of the support of  $C(t)$  (theoretically infinite but practically in the order of  $40\mu s$ ).

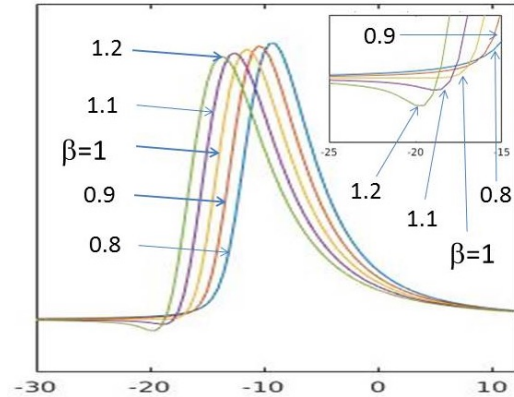


Figure 6: Probability densities of stable laws,  $b = 1.16$ ,  $c = 3.2$ ,  $m = 0$ .

## 4 Conclusion

Data in article [5] relate to attenuation and velocity of ultrasonics crossing polyethylene in the frequency band (4,10) MHz, at three temperatures. The attenuation follows a power law which places the problem in the Szabo's (causal) theory. Velocity measurements do not have the same accuracy and can be approached by different interpolation formulas. It is a situation often encountered [22], [23]. Velocity is a gently sloping line, where a small curvature is masked by inaccuracy of measurements.

In this paper, we explain data through random propagation times following stable probability laws. We give the values of probability laws parameters from the data of [5] and we explain why results are closely linked to causality.

## 5 Appendices

### 5.1 Appendix 1

In many situations (for instance propagation in atmosphere or water) a reasonable model for  $\mathbf{A}_x$ , the time spent on a length  $x$ , is a Gaussian process with mean  $mx$ , variance  $\sigma^2 x$  and autocorrelation function  $\sigma^2 x \rho(\tau)$  ( $\rho(0) = 1$ ) linear with  $x$ , taking into account the independence of delays on successive

pieces of the medium:

$$\begin{cases} \psi(f) = e^{-imfx - \sigma^2 f^2 x/2} \\ \phi(f, \tau) = e^{-\sigma^2 f^2 x(1-\rho(\tau))}. \end{cases} \quad (14)$$

Such hypothesis leads to Gaussian attenuation and constant velocity with respect to the frequency. If we are interested in the sound propagation in air at normal conditions, the values (in meters and seconds) in normal conditions

$$m = 0,003, \quad \sigma^2 = 4.10^{-11}$$

correspond to weakenings of an acoustic wave (at least for frequencies beyond  $10^5$ Hz). The equivalent probability density at  $x$  is defined by the Gaussian

$$C(t) = \frac{1}{\sigma\sqrt{2\pi x}} \exp \frac{-(t - mx)^2}{2\sigma^2 x}.$$

$\rho(\tau)$  in (14) measures the celerity of time variations of  $\mathbf{A}_1$ , which is linked to molecular motion. For a molecule of air ( $O_2$  or  $N_2$ ), the number of shocks by second is in the order of  $4.10^{10}$ , the time between two shocks in the order of  $3.10^{-10}$ s and the distance spent  $10^{-7}$ m. Obviously, these values have to lead to very high frequencies for  $\mathbf{V}_x$ . Furthermore, when  $x$  increases, it is the same for the power of  $\mathbf{V}_x$ , but it is likely that its spectral density flattens out, such that  $\mathbf{V}_x$  becomes less and less visible by devices matched to particular frequential windows.

## 5.2 Appendix 2: an example of random propagation time

Let the wave  $e^{2i\pi f_0 t}$  be crossing a 1 meter of ordinary air. The crossing time is about  $3.10^{-3}$ s. We ask the following question: if we assume that transit times are subject to small variations (for instance in the order of  $10^{-8}$ s.), is it possible to ignore them? Obviously, these small variations are not viewed by apparatus. Potential deformations will be attributed to noises due to receivers or external sources. Obviously, the question may be asked in reverse for a motionless medium and mobile receivers.

We consider a homogeneous Poisson process  $\{t_n, n \in \mathbb{Z}\}$  with parameter  $\lambda$  [11]. If  $N(t, \tau)$  is the number of  $t_n$  in  $(t, t + \tau)$ , we have the probabilities

$$P[N(t, \tau) = n] = \frac{(\lambda\tau)^n}{n!} e^{-\lambda\tau}.$$

where  $\lambda$  is the mean number of "times"  $t_n$  by time unit. Independently, we assume that the  $B_n$  are real, i.i.d (independent identically distributed) with c.f

$$E[e^{-2i\pi f B_n}] = \psi_0(f)$$

and we define the random propagation time  $\mathbf{A}$  by

$$A(t) = a + B_n, t_n \leq t < t_{n+1}.$$

where  $a$  is some positive quantity.  $B_n$  is a random gap added to the mean  $a$ . We firstly obtain ( $\tau > 0$ )

$$E[e^{-2i\pi f A(t)}] = \psi_0(f) e^{-2i\pi f a} = \psi(f)$$

$$\begin{aligned} \phi(f, \tau) &= E[e^{-2i\pi f(A(t) - A(t-\tau))}] \\ &= e^{-\lambda\tau} + |\psi(f)|^2 (1 - e^{-\lambda\tau}). \end{aligned}$$

In the decomposition (7)

$$e^{2i\pi f_0(t - A(t))} = \psi(f_0) e^{2i\pi f_0 t} + V(t)$$

the left hand term is the result of the monochromatic wave  $e^{2i\pi f_0 t}$  propagation.  $\psi(f_0) e^{2i\pi f_0 t}$  is the frequency line  $f_0$  after attenuation and delay defined by  $\psi(f_0)$ .  $V(t)$  is zero mean and verifies, from (8)

$$s_V(f) = \frac{2\lambda(1 - |\psi(f_0)|^2)}{4\pi^2(f - f_0)^2 + \lambda^2}$$

$s_V(f)$  is a power spectral density which is maximum at  $f = f_0$  with

$$s_V(f_0) = \frac{2}{\lambda} (1 - |\psi(f_0)|^2).$$

When  $\lambda$  is large enough, the "noise"  $\mathbf{V}$  will be invisible whatever the frequency window defined by the devices, though its total power is constant and equal to

$$P_V = \int_{-\infty}^{\infty} s_V(f) df = 1 - |\psi(f_0)|^2.$$

The choice of  $\psi(f)$  rules the value  $P_V$  (equivalently the attenuation), and the weight of  $\mathbf{V}$  in any frequency band can be adjusted by  $\lambda$ . As an example,  $\mathbf{A}$  is a Gaussian  $N(3.10^{-3}, 10^{-12})$ , in s. and  $s^2$ ,  $f_0 = 10^3 s^{-1}$ . We find  $P_V = 0.36$  and we have  $\lambda \sim 10^9$  with  $s_V(f) < 10^{-8}$ .

We can accept that it is this kind of model which represents the propagation in a medium made of a quasi infinite number of elements which interact.

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