Smooth Bias Estimation for Multipath Mitigation Using Sparse Estimation

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***Abstract*—Multipath remains the main source of error when using global navigation satellite systems (GNSS) in constrained environment, leading to biased measurements and thus to inaccurate estimated positions. This paper formulates the GNSS navigation problem as the resolution of an overdetermined system, which depends nonlinearly on the receiver position and linearly on the clock bias and drift, and possible biases affecting GNSS measurements. The extended Kalman filter is used to linearize the navigation problem whereas sparse estimation is considered to estimate multipath biases. We assume that only a part of the satellites are affected by multipath, i.e., that the unknown bias vector is sparse in the sense that several of its components are equal to zero. The natural way of enforcing sparsity is to introduce an ℓ_1 regularization associated with the bias vector. This leads to a least absolute shrinkage and selection operator (LASSO) problem that is solved using a reweighted-ℓ_1 algorithm. The weighting matrix of this algorithm is designed carefully as functions of the satellite carrier to noise density ratio and the satellite elevations. The smooth variations of multipath biases versus time are enforced using a regularization based on total variation. An experiment conducted on real data allows the performance of the proposed method to be appreciated.**

***Index Terms*—GNSS, multipath mitigation, sparse, LASSO, reweighted-ℓ_1 algorithm.**

I. Introduction

Multipath (MP) is one of the most difficult error sources that needs to be tackled for GNSS positioning [1]. MP signals are generally due to reflections on various obstacles, and thus strongly depend on the geometric configuration of the scene in which the receiver is located. More precisely, in the absence of obstacle, the receiver is not affected by MP. Conversely, when the receiver is located close to buildings, the received GNSS measurements are very likely to be subjected to MP. The problem of mitigating MP effects in GNSS measurements has received a considerable attention in the literature. MP can be mitigated at the antenna level [2] or at the receiver level, more precisely working on the correlator [3], [4] or the discriminator [5]. All these techniques require specific and expensive hardware that cannot always be purchased. Mitigating MP at a measurement or position level is thus an interesting alternative. A first solution is to take advantage of a 3D model of the environment to predict MP signals [6], and even to combine these techniques to other sensors, such as cameras. However, this 3D model is not always available in practical applications. A second option is to use the information available at the receiver, such as pseudoranges, Doppler shifts, satellite ephemeris and C/N_0, which is the carrier-to-noise density ratio (expressed in dB-Hz), corresponding to the ratio of the carrier power and the noise power per unit bandwidth [7]. Other techniques consist in exploiting different measurements from the same satellite, for instance the difference between the measurements from two receivers leading to differential GNSS [8, ch. 8] or even from two different users (collaborative or cooperative positioning) [9]. An interesting family of MP mitigation methods rely on statistical tests trying to exclude or correct the faulty measurements. The receiver autonomous integrity monitoring (RAIM) method belongs to this class of strategies [8, ch. 15]. More recent technique uses a-contrario modeling for discarding bad satellites [10]. Note that these techniques require redundant measurements, that are not always available in urban environment, and that the user will only be able to detect/estimate up to two faulty measurements. Other techniques based on sequential Monte Carlo methods, also referred to as particle filters, have been proposed in [11]. However, these methods are computationally intensive, making a real time implementation very complicated. Finally, it is interesting to mention other techniques based on non-Gaussian error terms, such as Markov processes [12] or Dirichlet process mixtures [13].

The point of view considered in this work is to model the effect of MP signals on GNSS measurements as sparse additive biases following the recent paper [14]. These biases are then estimated and subtracted from the GNSS measurements to mitigate MP effects. The bias estimation strategy is based on a reweighted-ℓ_1 algorithm projecting the observed measurements on an appropriate subspace related to the GNSS geometry matrix. However, we have observed that this estimation can be impacted by biases that are not in accordance with the a priori weights used for their estimation. The main contribution of this paper is to add a smooth regularization term to the reweighted-ℓ_1 algorithm of [14] enforcing smooth variations of the additive MP biases. This regularization is motivated by the fact that biases due to MP are generally observed at several consecutive time instants. Introducing smoothness in the LASSO problem of [15] by using a total variation regularization will result in a fused LASSO problem [16]. Note that we do not want the problem to be smooth in the

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satellite domain, since different satellite can be affected or not by MP depending on the satellite geometry.

This paper is organized as follows: Section II summarizes some basic principles on satellite navigation, describing how measurements (code measurements and Doppler rates) are related to the state vector (position, velocity) and the possible MP biases. This section also recalls the Kalman filtering steps that will be used to track the receiver position. Section III presents the method proposed in [14] to estimate MP biases using sparse estimation, formulating the positioning problem as a penalized least squares problem with a weighted ℓ_1 penalty. Section IV explains how the algorithm can be generalized to ensure smooth MP bias variations, avoiding loss of estimation in certain situations. Finally, Section V evaluates the performance of the different estimation methods via several experimental results, showing interesting improvements for local estimation problems as well as on the full validation campaign.

II. GNSS Navigation

A. State model

The GNSS navigation problem is formulated using the method described in [8, ch. 7], which is summarized in this section. The unknown state vector at time k (to be estimated) is defined as $\boldsymbol{X}_k = (x_k, y_k, z_k, b_k, \dot{x}_k, \dot{y}_k, \dot{z}_k, \dot{b}_k)^T$ where $\boldsymbol{r}_k = (x_k, y_k, z_k)^T$ and $\boldsymbol{v}_k = (\dot{x}_k, \dot{y}_k, \dot{z}_k)^T$ are the receiver position and velocity in a given frame, b_k is the receiver clock bias, $\dot{b}_k$ is the receiver clock drift, and the subscript k means that the variable or vector corresponds to time instant k. A random walk is adopted for the state propagation leading to

$$\boldsymbol{X}_{k+1} = \boldsymbol{F}_k \boldsymbol{X}_k + \boldsymbol{u}_k \quad \text{with} \quad \boldsymbol{F}_k = \begin{bmatrix} \boldsymbol{I}_4 & (\Delta t_k)\boldsymbol{I}_4 \\ \boldsymbol{0}_4 & \boldsymbol{I}_4 \end{bmatrix} \tag{1}$$

where $\boldsymbol{I}_4$ is the $\mathbb{R}^{4\times4}$ identity matrix, $\boldsymbol{0}_4$ is the $\mathbb{R}^{4\times4}$ zero matrix, Δt_k is the time between time instants k and $k+1$, and $\boldsymbol{u}_k$ is a zero-mean Gaussian noise vector of covariance matrix $\boldsymbol{Q}_k \in \mathbb{R}^{8\times8}$, i.e.,

$$\boldsymbol{u}_k \sim \mathcal{N}(\boldsymbol{0}_8, \boldsymbol{Q}_k) \tag{2}$$

where $\boldsymbol{0}_8$ is the zero vector of $\mathbb{R}^8$ and $\mathcal{N}(.)$ is the Gaussian distribution (closed-form expressions for $\boldsymbol{Q}_k$ can be found in [17, ch. 11], [18, ch. 12]).

B. Observation model

To estimate the unknown state vector $\boldsymbol{X}_k$, we assume that the receiver has access to two kinds of measurements: the pseudoranges related to the geometric distances between the receiver and the satellites, and the pseudorange rates, which are the Doppler measurements up to a multiplicative constant, related to the relative velocities between the receiver and the satellites. Denote as s_k the number of satellites visible at time instant k. The number of measurements acquired by the receiver is $2s_k$, namely s_k pseudoranges denoted as $\rho_{1,k}, ..., \rho_{s_k,k}$ and s_k pseudorange rates, denoted as $\dot{\rho}_{1,k}, ..., \dot{\rho}_{s_k,k}$. These measurements are gathered in the vector $\boldsymbol{z}_k = (z_{k,1}, ..., z_{k,2s_k})^T \in \mathbb{R}^{2s_k}$ whose components are

$$z_{k,i} = \rho_{i,k} \quad \text{and} \quad z_{k,i+s_k} = \dot{\rho}_{i,k} \tag{3}$$

for $i = 1, ..., s_k$. As mentioned before, these measurements are functions of the various components of the state vector. More precisely, using the notations $\boldsymbol{r}_{i,k} = (x_{i,k}, y_{i,k}, z_{i,k})^T$ and $\boldsymbol{v}_{i,k} = (\dot{x}_{i,k}, \dot{y}_{i,k}, \dot{z}_{i,k})^T$ for the i-th satellite position and velocity at time instant k, we obtain

$$\rho_{i,k} = \|\boldsymbol{r}_k - \boldsymbol{r}_{i,k}\|_2 + b_k + \varepsilon_{i,k} \tag{4}$$

$$\dot{\rho}_{i,k} = (\boldsymbol{r}_k - \boldsymbol{v}_{i,k})^T \frac{\boldsymbol{r}_k - \boldsymbol{r}_{i,k}}{\|\boldsymbol{r}_k - \boldsymbol{r}_{i,k}\|_2} + \dot{b}_k + \dot{\varepsilon}_{i,k} \tag{5}$$

where

- $\boldsymbol{r}_{i,k} = (x_{i,k}, y_{i,k}, z_{i,k})^T$ is the i-th satellite position at time k expressed in the same frame as $\boldsymbol{r}_k$,
- $\boldsymbol{v}_{i,k} = (\dot{x}_{i,k}, \dot{y}_{i,k}, \dot{z}_{i,k})^T$ is the ith satellite velocity at time instant k expressed in the same frame as $\boldsymbol{v}_k$,
- $\|\boldsymbol{r}_k - \boldsymbol{r}_{i,k}\|_2 = \sqrt{(x_k - x_{i,k})^2 + (y_k - y_{i,k})^2 + (z_k - z_{i,k})^2}$ is the geometric distance between the user and the ith satellite,
- $\varepsilon_{i,k}$ and $\dot{\varepsilon}_{i,k}$ are the error terms associated with the ith propagation channel (modeling ionospheric delay, tropospheric delay, satellite clock bias, satellite position uncertainty, Sagnac effect, relativistic effects, MP and receiver noise).

Note that b_k and $\dot{b}_k$ affect the s_k corresponding measurements similarly. Usually, MP and receiver noise are treated apart from other error sources, for which we have models. We will assume that these models are sufficient for correcting the corresponding errors. Thus, the measurement equations can be rewritten as

$$\boldsymbol{z}_k = \boldsymbol{h}_k(\boldsymbol{X}_k) + \boldsymbol{m}_k + \boldsymbol{n}_k \tag{6}$$

where $\boldsymbol{m}_k = (m_{1,k}, ..., m_{2s_k,k})^T \in \mathbb{R}^{2s_k}$ is the vector accounting for the eventual MP biases, $\boldsymbol{n}_k = (n_{1,k}, ..., n_{2s_k,k})^T \in \mathbb{R}^{2s_k}$ is the receiver noise vector supposed centered and Gaussian with covariance matrix $\boldsymbol{R}_k \in \mathbb{R}^{2s_k \times 2s_k}$ defined as

$$\boldsymbol{R}_k = \begin{bmatrix} \sigma^2_{\text{UERE}} I_{s_k} & \boldsymbol{0}_{s_k \times s_k} \\ \boldsymbol{0}_{s_k \times s_k} & \dot{\sigma}^2_{\text{UERE}} I_{s_k} \end{bmatrix} \tag{7}$$

where σ^2_{UERE} and $\dot{\sigma}^2_{\text{UERE}}$ are constants used for the pseudorange and pseudorange rate variances (UERE stands for "User Equivalent Range Error" [8, p. 208]), and $\boldsymbol{h}_k$ is a nonlinear function which is not provided here but can be deduced from (4), (5) and error models as in [8, ch. 7]. Given the linear state equation (1) and the non-linear measurement equation (6), it is natural to investigate the extended Kalman filter (EKF) [8, ch. 3], [19, ch. 8] to estimate the state vector $\boldsymbol{X}_k$.

C. The extended Kalman filter for navigation

In absence of additive bias $\boldsymbol{m}_k$, i.e., for centered errors in (6) (such that $\boldsymbol{m}_k = 0$), the EKF leads to

$$\hat{\boldsymbol{X}}_{k|k-1} = \boldsymbol{F}_{k-1}\hat{\boldsymbol{X}}_{k-1|k-1} \tag{8}$$

$$\boldsymbol{P}_{k|k-1} = \boldsymbol{F}_{k-1}\boldsymbol{P}_{k-1|k-1}\boldsymbol{F}_{k-1}^T + \boldsymbol{Q}_{k-1} \tag{9}$$

$$\boldsymbol{K}_k = \boldsymbol{P}_{k|k-1}\boldsymbol{H}_k^T\left(\boldsymbol{H}_k\boldsymbol{P}_{k|k-1}\boldsymbol{H}_k^T + \boldsymbol{R}_k\right)^{-1} \tag{10}$$

$$\hat{\boldsymbol{X}}_{k|k} = \hat{\boldsymbol{X}}_{k|k-1} + \boldsymbol{K}_k(\boldsymbol{z}_k - \boldsymbol{h}_k(\hat{\boldsymbol{X}}_{k|k-1})) \tag{11}$$

$$\boldsymbol{P}_{k|k} = (I_8 - \boldsymbol{K}_k\boldsymbol{H}_k)\boldsymbol{P}_{k|k-1} \tag{12}$$

where $\boldsymbol{H}_k \in \mathbb{R}^{2s_k\times 8}$ is the Jacobian matrix of the function $\boldsymbol{h}_k$ at point $\hat{\boldsymbol{X}}_{k|k-1}$. This filter consists in applying a Kalman filter to the state equation (1) and the first order approximation around $\hat{\boldsymbol{X}}_{k|k-1}$ of $\boldsymbol{h}_k(\boldsymbol{X}_k)$ in (6) with $\boldsymbol{m}_k = 0$

$$\boldsymbol{z}_k \approx \boldsymbol{h}_k(\hat{\boldsymbol{X}}_{k|k-1}) + \boldsymbol{H}_k(\boldsymbol{X}_k - \hat{\boldsymbol{X}}_{k|k-1}) + \boldsymbol{n}_k. \tag{13}$$

If $\boldsymbol{m}_k$ would be a known bias term, the EKF would be similar to the above derivations, except that (11) should be replaced in the previous filter by

$$\hat{\boldsymbol{X}}_{k|k} = \hat{\boldsymbol{X}}_{k|k-1} + \boldsymbol{K}_k(\boldsymbol{z}_k - \boldsymbol{h}_k(\hat{\boldsymbol{X}}_{k|k-1}) - \boldsymbol{m}_k). \tag{14}$$

The next section proposes a method allowing the unknown vector $\boldsymbol{m}_k$ to be estimated using some sparsity constraints. The estimated vector will be used in place of $\boldsymbol{m}_k$ in (14) to mitigate MP effects.

III. Sparse estimation theory applied to GNSS multipath mitigation

This section recalls the principles of the sparse estimation method of [14] using the LASSO algorithm and a reweighted-ℓ_1 regularization.

A. The LASSO problem

Assume that we have a vector of measurements $\tilde{\boldsymbol{y}}_k \in \mathbb{R}^{2s_k}$ defined as $\tilde{\boldsymbol{y}}_k = \tilde{\boldsymbol{H}}_k\boldsymbol{\theta}_k + \tilde{\boldsymbol{n}}_k$, where $\tilde{\boldsymbol{H}}_k \in \mathbb{R}^{2s_k\times 2s_k}$ is a known regression matrix, $\boldsymbol{\theta}_k \in \mathbb{R}^{2s_k}$ is an unknown sparse vector (to be estimated) and $\tilde{\boldsymbol{n}}_k \in \mathbb{R}^{2s_k}$ is an unknown error term[1]. A classical way of estimating $\boldsymbol{\theta}_k$ from the observed measurement vector $\tilde{\boldsymbol{y}}_k$ is to consider a data fidelity term $\frac{1}{2}\|\tilde{\boldsymbol{y}}_k - \tilde{\boldsymbol{H}}_k\boldsymbol{\theta}_k\|_2^2$ penalized by an additive regularization promoting the sparsity of $\boldsymbol{\theta}_k$ as the problem is underdetermined (when $\tilde{\boldsymbol{H}}_k$ is not full rank). One can think of defining this additive regularization as the ℓ_1 norm of $\boldsymbol{\theta}_k$ defined by

$$\|\boldsymbol{\theta}_k\|_1 = \sum_{i=1}^{2s_k}|\theta_{k,i}|. \tag{15}$$

This problem formulation leads to the so-called LASSO estimator defined as [15]

$$\hat{\boldsymbol{\theta}}_k = \underset{\boldsymbol{\theta}_k\in\mathbb{R}^{2s_k}}{\arg\min}\frac{1}{2}\|\tilde{\boldsymbol{y}}_k - \tilde{\boldsymbol{H}}_k\boldsymbol{\theta}_k\|_2^2 + \lambda_k\|\boldsymbol{\theta}_k\|_1 \tag{16}$$

where $\lambda_k \in \mathbb{R}^+$ is a fixed constant referred to as regularization parameter.

[1]The meaning of the different vectors $\tilde{\boldsymbol{y}}_k, \boldsymbol{\theta}_k, \tilde{\boldsymbol{n}}_k$ in the GNSS context will be clarified in subsection III-C.

B. The reweighted-$\ell_1$algorithm of [14]

Candès [20] investigated a so-called reweighted-ℓ_1 method defined as follows

$$\underset{\boldsymbol{\theta}_k\in\mathbb{R}^{2s_k}}{\arg\min}\frac{1}{2}\|\tilde{\boldsymbol{y}}_k - \tilde{\boldsymbol{H}}_k\boldsymbol{\theta}_k\|_2^2 + \lambda\|\boldsymbol{W}_k\boldsymbol{\theta}_k\|_1 \tag{17}$$

where $\boldsymbol{W}_k \in \mathbb{R}^{2s_k\times 2s_k}$ is a diagonal weighting matrix. Ideally, the weights contained in $\boldsymbol{W}_k$ should be inversely proportional to the magnitude of the true unknown vector $\boldsymbol{\theta}_0$, i.e., such that

$$w_{k,i} = \begin{cases} \frac{1}{|\theta_{0,i}|}, & \theta_{0,i} \neq 0, \\ \infty, & \theta_{0,i} = 0. \end{cases} \tag{18}$$

However, this weight definition cannot be used in practice since $\boldsymbol{\theta}_0$ is an unknown vector. An iterative solution was proposed in [20], but did not give good results for our application. Looking carefully at (18), we can see that if we know a priori that $\theta_{0,i}$ has a large (resp. small) value, we should define a low (resp. high) weight $w_{k,i}$. The weighting strategy proposed in the next section precisely meets this property.

C. A reweighted-ℓ_1 method for GNSS

In the presence of an additive bias affecting the measurement equation, using the notations $\boldsymbol{y}_k = \boldsymbol{z}_k - \boldsymbol{h}_k(\hat{\boldsymbol{X}}_{k|k-1}) \in \mathbb{R}^{2s_k}$ and $\boldsymbol{x}_k = \boldsymbol{X}_k - \hat{\boldsymbol{X}}_{k|k-1}$, Eq. (13) should be rewritten

$$\boldsymbol{y}_k = \boldsymbol{H}_k\boldsymbol{x}_k + \boldsymbol{m}_k + \boldsymbol{n}_k. \tag{19}$$

The proposed MP mitigation method assumes that the bias vector $\boldsymbol{m}_k$ is sparse, i.e., that some of its components are exactly equal to 0. Exploiting this sparsity assumption, and following the previous ideas, we propose to solve the following problem

$$\underset{\boldsymbol{x}_k,\boldsymbol{m}_k}{\arg\min}\frac{1}{2}\|\boldsymbol{y}_k - \boldsymbol{H}_k\boldsymbol{x}_k - \boldsymbol{m}_k\|_2^2 + \lambda_k\|\boldsymbol{W}_k\boldsymbol{m}_k\|_1 \tag{20}$$

in order to detect the measurements affected by MP (i.e., measurements affected by the presence of additive biases), estimate the corresponding biases, and replace them in (19). Regarding the weighting matrix $\boldsymbol{W}_k$, we propose to consider the strategy of [14], leading to

$$w_1(x) = \tag{21}$$
$$\begin{cases} 10^{\frac{x-T}{a}}\left(\left(A\times 10^{\frac{F-T}{a}} - 1\right)\frac{x-T}{F-T} + 1\right)^{-1}, & x < T \\ 1, & x \geq T \end{cases}$$

where

- x is the value of C/N_0 expressed in dBHz,
- $T = 45$ is a threshold after which the weight is set to 1 (indicating that the measurements are "good"),
- $a = 80$ allows the bending of the curve to be adjusted,
- $F = 20$ defines the value of C/N_0 for which the function w_1 is forced to have the weight defined by parameter A
- $A = 30$ controls the value of the function w_1 for $x = F$ ($w_1(F) = 1/A$).

and

$$w_2(x) = \begin{cases} \frac{\sin^2(x)}{\sin^2(5^\circ)} & x < 5^\circ \\ 1 & x \geq 5^\circ \end{cases} \quad (22)$$

where x is a given satellite elevation (also referred to as altitude) expressed in degrees. The final weight introduced in the reweighted-ℓ_1 approach is defined as the product of the two previous functions for each satellite, i.e.,

$$w_{i,k}\left[(C/N_0)_{i,k}, e_{i,k}\right] = w_1\left[(C/N_0)_{i,k}\right] w_2(e_{i,k}) \quad (23)$$

where $w_{i,k}$ is the i-th diagonal element of the matrix $\boldsymbol{W}_k$, $(C/N_0)_{i,k}$ and $e_{i,k}$ are the C/N_0 and elevation associated with the ith satellite at time instant k. These weights give more importance to satellites associated with high C/N_0 and/or elevation. Conversely low weights are assigned to the satellites with low elevation and/or C/N_0, since these satellites are more likely to suffer from MP.

In order to obtain a formulation similar to (17), it is interesting to note that the minimization of (20) with respect to $\boldsymbol{x}_k$ for a fixed $\boldsymbol{m}_k$ has the following closed-form expression

$$\boldsymbol{x}_k = (\boldsymbol{H}_k^T \boldsymbol{H}_k)^{-1} \boldsymbol{H}_k^T (\boldsymbol{y}_k - \boldsymbol{m}_k) \quad (24)$$

which is the classical least squares solution. After replacing this expression of $\boldsymbol{x}_k$ in (20), we obtain the so-called profile likelihood

$$L(\boldsymbol{m}_k) = \frac{1}{2}\|(I_{2s_k} - \boldsymbol{P}_k)(\boldsymbol{y} - \boldsymbol{m}_k)\|_2^2 + \lambda_k \|\boldsymbol{W}_k \boldsymbol{m}_k\|_1 \quad (25)$$

where $\boldsymbol{P}_k$ is the following projection matrix

$$\boldsymbol{P}_k = \boldsymbol{H}_k (\boldsymbol{H}_k^T \boldsymbol{H}_k)^{-1} \boldsymbol{H}_k^T. \quad (26)$$

Finally, after introducing the following notations

$$\tilde{\boldsymbol{y}}_k = (I_{2s_k} - \boldsymbol{P}_k)\boldsymbol{y}_k \quad (27)$$

$$\tilde{\boldsymbol{H}}_k = (I_{2s_k} - \boldsymbol{P}_k)\boldsymbol{W}_k^{-1} \quad (28)$$

$$\boldsymbol{\theta}_k = \boldsymbol{W}_k \boldsymbol{m}_k \quad (29)$$

the original problem (20) reduces to

$$\underset{\boldsymbol{\theta}_k \in \mathbb{R}^{2s_k}}{\arg\min} \frac{1}{2}\|\tilde{\boldsymbol{y}}_k - \tilde{\boldsymbol{H}}_k \boldsymbol{\theta}_k\|_2^2 + \lambda_k \|\boldsymbol{\theta}_k\|_1. \quad (30)$$

As a consequence, we have to solve a LASSO problem whose solution can be obtained using classical efficient algorithms [15], [21], [22]. In this paper, we have used the "shooting algorithm" (detailed for instance in [23] and [24]). The resulting MP mitigation strategy can be summarized as follows

1) estimate the unknown parameter vector $\boldsymbol{\theta}_k$ as the solution of the LASSO problem (30) yielding $\hat{\boldsymbol{\theta}}_k$,
2) estimate the bias vector as $\hat{\boldsymbol{m}}_k = \boldsymbol{W}_k^{-1}\hat{\boldsymbol{\theta}}_k$,
3) consider the EKF proposed in Section II-C defined by (8), (9), (10), (14) and (12).

This method has shown interesting results in many practical scenarios [14]. However, we have observed some problems when the proposed weighting is not in agreement with the actual bias values, e.g., when the a priori weight is high (which means that it is a priori not likely to have an MP bias on the considered satellite), and there is an important bias affecting the observed measurements. In such cases, we have observed that even if the algorithm fails, the biases at the previous time instants were estimated correctly, which could be exploited to improve estimation performance. Therefore, we propose to introduce some temporal smoothness for successive bias estimates. The next section describes the strategy adopted to introduce this temporal smoothness.

IV. Smooth sparse estimation

Jointly imposing some sparsity and smoothness properties is the objective of the fused LASSO described in [16] and defined as

$$\underset{\boldsymbol{\theta}_k \in \mathbb{R}^{2s_k}}{\arg\min} \frac{1}{2}\|\tilde{\boldsymbol{y}}_k - \tilde{\boldsymbol{H}}_k \boldsymbol{\theta}_k\|_2^2 + \lambda_k \|\boldsymbol{\theta}_k\|_1 + \mu_k \sum_{i=2}^{2s_k} |\theta_{k,i} - \theta_{k,i-1}|. \quad (31)$$

where the last regularization term is referred to as total variation (TV) and $\mu_k \in \mathbb{R}^+$ is a regularization parameter. If we keep the previous equation, the smoothness will be induced along satellites, which is not our main objective. In order to ensure a temporal smoothness, we propose to introduce a penalty associated with the temporal variations of the different biases leading to the following problem

$$\underset{\boldsymbol{\theta}_k \in \mathbb{R}^{2s_k}}{\arg\min} \frac{1}{2}\|\tilde{\boldsymbol{y}}_k - \tilde{\boldsymbol{H}}_k \boldsymbol{\theta}_k\|_2^2 + \lambda_k \|\boldsymbol{\theta}_k\|_1 + \mu_k \sum_{i=1}^{2s_k} |\theta_{k,i} - \hat{\theta}_{k-1,i}| \quad (32)$$

or equivalently

$$\underset{\boldsymbol{\theta}_k \in \mathbb{R}^{2s_k}}{\arg\min} \frac{1}{2}\|\tilde{\boldsymbol{y}}_k - \tilde{\boldsymbol{H}}_k \boldsymbol{\theta}_k\|_2^2 + \lambda_k \|\boldsymbol{\theta}_k\|_1 + \mu_k \|\boldsymbol{\theta}_k - \hat{\boldsymbol{\theta}}_{k-1}\|_1. \quad (33)$$

Note that the temporal smoothing is assigned to the weighted biases and not to the biases themselves. Indeed, this strategy induces more smoothing to channels affected by large weights, which is a desired property.

However, some satellites might not be visible at some time instants k. Thus, the last regularizer has to be only evaluated for satellites that are visible at time instants k and $k-1$. In order to respect this constraint, we introduce the following penalty

$$\|\boldsymbol{\theta}_k - \hat{\boldsymbol{\theta}}_{k-1}\|_{1,\mathcal{S}_k} = \sum_{i \in \mathcal{S}_k} |\theta_{i,k} - \hat{\theta}_{i,k-1}| \quad (34)$$

where $\mathcal{S}_k$ is the set of indices associated with satellites that are jointly visible at time instants k and $k-1$. Therefore we propose to modify the MP mitigation technique of [14] as

1) estimate the unknown parameter vector $\hat{\boldsymbol{\theta}}_k$ as the solution of

$$\underset{\boldsymbol{\theta}_k \in \mathbb{R}^{s_k}}{\arg\min} \frac{1}{2}\|\tilde{\boldsymbol{y}}_k - \tilde{\boldsymbol{H}}_k \boldsymbol{\theta}_k\|_2^2 + \lambda_k \|\boldsymbol{\theta}_k\|_1 + \mu_k \|\boldsymbol{\theta}_k - \hat{\boldsymbol{\theta}}_{k-1}\|_{1,\mathcal{S}_k} \quad (35)$$

2) estimate the bias vector as $\hat{\boldsymbol{m}}_k = \boldsymbol{W}_k^{-1}\hat{\boldsymbol{\theta}}_k$,

3) consider the EKF proposed in Section II-C with equations (8),(9),(10),(14) and (12).

Remarks:

1) *we will also consider an ℓ_2 smoothing penalty term, replacing the ℓ_1 norm in Step 1) of the previous algorithm by an ℓ_2 norm, leading to*

$$\underset{\boldsymbol{\theta}_k \in \mathbb{R}^{s_k}}{\arg\min} \frac{1}{2}\|\tilde{\boldsymbol{y}}_k - \tilde{\boldsymbol{H}}_k\boldsymbol{\theta}_k\|_2^2 + \lambda_k\|\boldsymbol{\theta}_k\|_1 + \mu_k\|\boldsymbol{\theta}_k - \hat{\boldsymbol{\theta}}_{k-1}\|_{2,\mathcal{S}_k}^2 \tag{36}$$

where $\|.\|_{2,\mathcal{S}_k}$ is defined as $\|.\|_{1,\mathcal{S}_k}$ with an ℓ_2-norm.

2) *To solve the problem with the ℓ_2-smoothing penalty of* (36)*, we introduce the notation $\boldsymbol{\Delta}_k = diag(i \in \mathcal{S}_k)$, which is the diagonal matrix whose i-th entry is 1 if the corresponding satellite was visible at the previous time instant, and 0 if it was not. Thus, the problem* (36) *can be rewritten*

$$\underset{\boldsymbol{\theta}_k \in \mathbb{R}^{s_k}}{\arg\min} \frac{1}{2}\|\tilde{\boldsymbol{y}}_k - \tilde{\boldsymbol{H}}_k\boldsymbol{\theta}_k\|_2^2 + \lambda_k\|\boldsymbol{\theta}_k\|_1 + \mu_k\|\boldsymbol{\Delta}_k(\boldsymbol{\theta}_k - \hat{\boldsymbol{\theta}}_{k-1})\|_2^2. \tag{37}$$

Denote as $\boldsymbol{A}_k$ the square root matrix of $\tilde{\boldsymbol{H}}_k^T\tilde{\boldsymbol{H}}_k + 2\mu_k\boldsymbol{\Delta}_k$ such that $\boldsymbol{A}_k^T\boldsymbol{A}_k = \tilde{\boldsymbol{H}}_k^T\tilde{\boldsymbol{H}}_k + 2\mu_k\boldsymbol{\Delta}_k$. Straightforward computations allow the previous problem to be rewritten as

$$\underset{\boldsymbol{\theta}_k \in \mathbb{R}^{s_k}}{\arg\min} \frac{1}{2}\|(\boldsymbol{A}_k^T)^{-1}(\tilde{\boldsymbol{H}}_k^T\tilde{\boldsymbol{y}}_k + \mu_k\boldsymbol{\Delta}_k\hat{\boldsymbol{\theta}}_{k-1}) - \boldsymbol{A}_k\boldsymbol{\theta}_k\|_2^2 + \lambda_k\|\boldsymbol{\theta}_k\|_1 \tag{38}$$

which is a LASSO problem.

3) *To solve the problem with the ℓ_1-smoothing term, a method similar to the one proposed in [24] can be used to derive the corresponding shooting algorithm. More details can be found in the technical report [25].*

The next section evaluates the performance of the proposed GNSS navigation strategy and compares it with the method proposed in [14].

V. Experimental results

To appreciate the efficiency of the proposed method, it was evaluated on real measurements provided by a u-blox AEK-4T receiver, and compared with the solution investigated in [14] (without smoothness), and the solution associated with the $\|.\|_{2,\mathcal{S}_k}$ penalty. A reference solution was obtained during the measurement campaign using a very accurate (high-cost) receiver, i.e., a Novatel SPAN composed of a GPS receiver Propak-V3 and an inertial measurement unit (IMAR). Figure 1 displays the situation that led us to study the proposed method, showing the theoretical and estimated pseudorange rate bias and C/N_0 for satellite #13 versus time. As can be seen, between time instants 313220 and 313225 seconds, the important bias cannot be mitigated leading to the bad cyan positions shown in Fig. 2 (the green arrow shows the instant from which the bias is badly estimated). Fig. 2 also shows the estimates obtained with the $\|.\|_{2,\mathcal{S}_k}$ smooth regularization and the proposed solution providing very competitive results. Note that the reduced performance obtained with the method of [14] is due to a disagreement between the high theoretical bias and the high value of C/N_0 (> 40 dBHz). More precisely, due to the high value of C/N_0, the weight in the reweighted ℓ_1 algorithm is large, preventing the bias to be estimated.

The different methods were also evaluated during the full campaign described in [14]. The corresponding cumulative distribution functions (CDFs) are displayed in Fig. 3. Fig. 3 clearly shows that the planar (horizontal) and altitude (vertical) errors obtained with the proposed ℓ_1-smoothed method are globally smaller than those obtained with the other methods. The effects of the temporal smoothing introduced for the biases tend to correct local positioning errors such as the one observed in Fig. 2 indicated by the green circle.

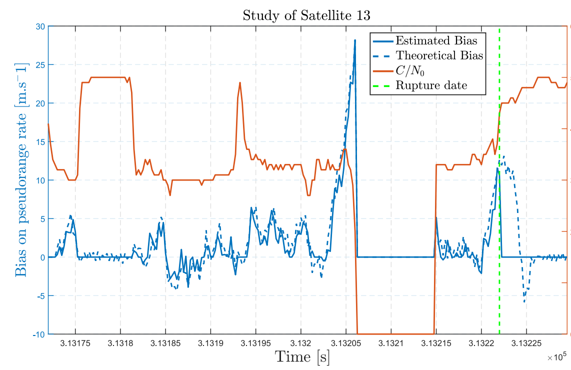

Fig. 1: Typical example showing an MP bias that has not been mitigated. The figure displays the estimated and theoretical biases for the pseudorange rate of satellite #13 (in blue) and the C/N_0 values (in orange) versus time.

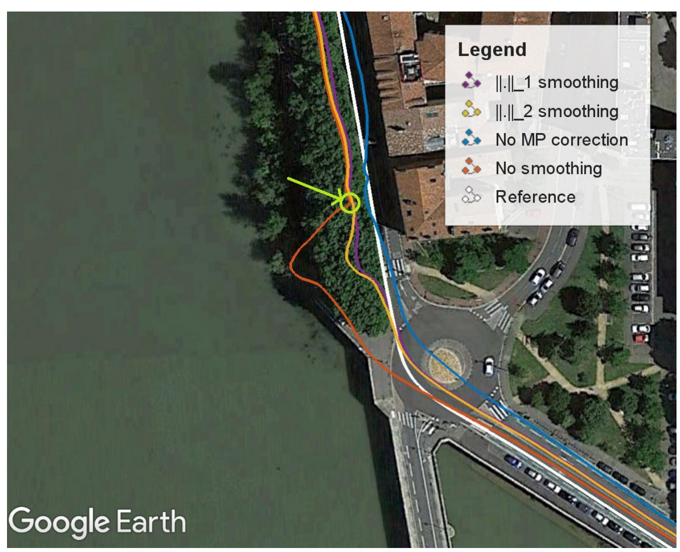

Fig. 2: Actual and estimated trajectories: reference (white), without MP correction (blue), reweighted-ℓ_1 (referred to as "No smoothing") (red), smooth-ℓ_2 reweighted-ℓ_1 (yellow), smooth-ℓ_1 (purple), and a specific change instant (green circle). Note that the vehicle moved from top to bottom.

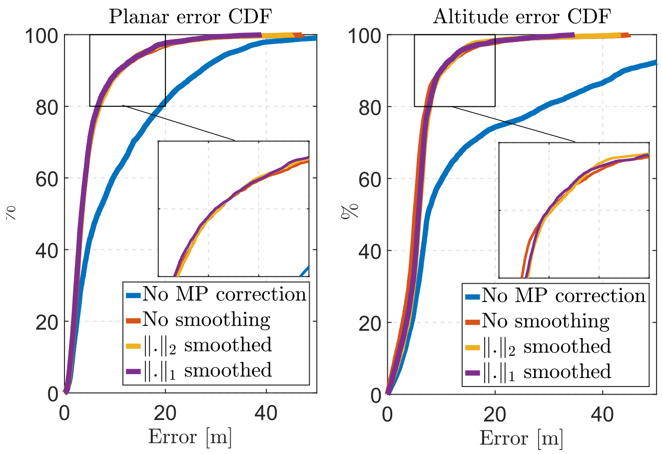

Fig. 3: Planar (left) and altitude (right) error CDFs for the different methods.

VI. Discussion

The method introduced in this paper was motivated by the results obtained in [14]. Indeed, even if the global performance of the proposed multipath mitigation method was quite promising, we observed some local problems due to a wrong estimation of biases introduced in the navigation model. An example of problem was displayed in Fig. 2 showing the estimated trajectories obtained without (blue curve) and with (red curve) bias estimation/correction resulting from the reweighted-ℓ_1 method of [14]. These problems were due to discontinuities in the estimated biases corresponding to a channel with a high value of C/N_0 but affected by multipath. Therefore, we had the idea of introducing a smoothing step for channels characterized by high values of C/N_0 or high elevations (i.e., channels with high weights). This smoothing step provided very interesting results with better estimated trajectories (as the purple one shown in Fig. 2). The multipath mitigation investigated in this paper allowed local estimation problems to be corrected with a global estimation performance equivalent to the one obtained with the method in [14] (as displayed in Fig. 3).

VII. Conclusions

This paper investigated a modification of the reweighted-ℓ_1 method investigated in [14] to mitigate multipath effects for GNSS navigation. The proposed modified algorithm exploited the joint smoothness and sparsity properties of MP affecting the different satellite channels. Experiments conducted on real data clearly outlined the benefits of including a temporal bias smoothness. One possible investigation for future work is to consider a more general distribution for the measurement noise, which was assumed white Gaussian with a common variance for all the pseudoranges and a common variance for all the pseudorange rates. Note that some estimation methods such as the well known weighted least squares algorithm aim at weighting the measurements according to the noise variances [26], [27], which would deserve to be investigated for our problem.

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