LLR Approximation for Fading Channels Using a Bayesian Approach

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***Abstract*— This letter investigates on the derivation of good log likelihood ratio (LLR) approximations under uncorrelated fading channels with partial statistical channel state information (CSI) at the receiver. While previous works focused mainly on solutions exploiting full statistical CSI over the normalized Rayleigh fading channel, in this letter, a Bayesian approach based on conjugate prior analysis is proposed to derive LLR values that only uses moments of order one and two associated with the random fading coefficients. The proposed approach is shown to be a more robust method compared to the best existing approximations, since it can be performed independently of the fading channel distribution and, in most cases, at a lower complexity. Results are validated for both binary and M-ary modulations over different uncorrelated fading channels.**

***Index Terms*— LLR values, fading channel, channel uncertainty, M-ary modulations, best linear approximation.**

I. Introduction

IN MODERN error correcting algorithms, the input of the associated soft input decoding algorithms mainly relies on the so-called log likelihood ratio (LLR) values [1], [2]. These LLR values can be shown to be sufficient statistics for the decoding and detection process. Typically, in order to compute a closed-form of these LLR values, the knowledge of the propagation channel, referred to as complete channel state information (CSI) is assumed, i.e. the channel is perfectly known. However, this assumption can be untrue in real applications since the complete CSI might be not fully available at the receiver [3]. In this work, we focus on uncorrelated fading channels with binary and non-binary inputs, modeled with a fading gain h and an additive Gaussian noise $w_n \sim \mathcal{N}(0, \sigma^2)$. If h and σ^2 are known at the receiver (complete CSI case) and a binary modulation is used, the LLRs can be computed as a linear function of the channel output [4], [5]. However for non-binary modulations, LLRs are non linear functions of the channel output [6], increasing the receiver complexity. In order to handle this complexity, approximate LLRs have been previously proposed in the literature (e.g. [7]). If h cannot be precisely known and only full statistical CSI is available (i.e. we have the knowledge of the probability density function (pdf) associated with the fading coefficients), one is still able to derive a closed-form for LLR values, which is in general a non linear function of the channel output. To lower the complexity, several authors (see for example [8], [9] for binary-phase shift-keying (BPSK) modulation and [6], [9] for M-ary modulations) have proposed LLR approximations. Regarding to the BSPK case, [8] proposed an efficient method for which the best linear approximation can be shown to maximize an approximate mutual information based functional assuming that full statistical CSI is available, i.e. we can have access to the (conditional) pdf of the estimated LLRs. This work has been then extended to non-binary modulations in [6]. Another approach to compute analytically closed-form LLR approximations through the Taylor series was proposed in [9]. Thanks to this approach, it was possible to reduce the complexity issue for the method presented in [6]. However, those approximations are only available for the normalized Rayleigh/Rice distribution, for which an easy-to-handle closed-form of the derivative is available, which is not always possible in the general case.

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In this letter, we propose a different method following a Bayesian approximation based approach using conjugate prior analysis [10]. This method allows to derive simple analytical closed-form expressions of the LLR values, considering that only a partial statistical CSI (first and second moments of the fading gain) is available at the receiver. Since a conjugate prior is selected as a prior distribution for the fading gain, this method can be applied independently of the channel fading distribution $p(h)$. Moreover, considering that a learning sequence is available at the receiver, the first and the second statistical moments can be easily estimated based on state-of-the-art estimation techniques.

This letter is organized as follows: Section II reviews LLR expressions under complete CSI and full statistical CSI. In Section III, we present a novel Bayesian approach for the derivation of the LLR values when only partial statistical CSI is available. Moreover, we briefly present the online estimation of the parameters μ_h and σ_h^2 considering that a learning sequence is available at the receiver. Results are analyzed for two kinds of uncorrelated fading channels in Section IV. Conclusions and perspectives are finally drawn in Section V.

II. LLR Under Complete and Full Statistical CSI

Following [9], we consider an uncorrelated fading channel where the received signal is expressed as :

$$y_n = h_n \cdot x_n + w_n \tag{1}$$

where x_n and y_n represent the channel input and output at symbol time n, respectively; w_n is a zero mean (possibly complex) additive white Gaussian noise (AWGN) with variance σ^2

($2\sigma^2$ for bi-dimensional constellations), and h_n are the channel gains that are independent and identically distributed (i.i.d.) random variables with associated probability density function (pdf) given by $p(h)$, i.e. $h_n \sim p(h)$. We further assume that x_n and w_n are i.i.d. random variables (r.v.).

At the transmitter, we assume a bit-interleaved coded modulation (BICM) scheme where the binary information sequence $\mathbf{u} = [u_1, \ldots, u_K]$ is first encoded using a binary error correcting code of rate $R = K/N$, yielding a binary codeword $\mathbf{c} = [c_1, \ldots, c_N]$ of length $N > K$. Then, $\mathbf{c}$ is bit interleaved and divided into N_s blocks of m bits. $\forall k = 1, \ldots, N_s$, each block $\mathbf{b}_k = [b_k^1, \ldots, b_k^m]$ is mapped into a symbol x_k from an M-ary signal constellation $\mathcal{X}$ of size $|\mathcal{X}| = 2^m$. We further assume that Gray mapping is used. At the receiver, LLRs are computed for each interleaved bit and then used to feed the input of the soft channel decoder. For the case of complete CSI (i.e. h_n is perfectly known), the LLR associated with the i-th transmitted bit $b_n^i(x_n)$, $i = 1, \ldots, m$, associated with the n-th transmitted symbol $x_n \in \mathcal{X}$ is given by

$$\mathcal{L}_n^{(i)} = \ln\left(\frac{P(y_n|b_n^i(x_n) = 0, h_n)}{P(y_n|b_n^i(x_n) = 1, h_n)}\right)$$

$$= \ln\left(\frac{\sum_{x_n \in \mathcal{X}_0(i)} P(y_n|x_n, h_n)}{\sum_{x_n \in \mathcal{X}_1(i)} P(y_n|x_n, h_n)}\right) \quad (2)$$

where $\mathcal{X}_j(i)$ is the subset of symbols of $\mathcal{X}$ where $b_n^i(x_n) = j$, $j \in \{0, 1\}$. When h_n cannot be perfectly known at the receiver, but $p(h)$ is known (full statistical CSI case) as a prior knowledge, the LLR expression can be computed as

$$\mathcal{L}_n^{(i)} = \ln\left(\frac{\sum_{x_n \in \mathcal{X}_0(i)} \int_{-\infty}^{\infty} P(y_n|x_n, h)p(h)\, dh}{\sum_{x_n \in \mathcal{X}_1(i)} \int_{-\infty}^{\infty} P(y_n|x_n, h)p(h)\, dh}\right). \quad (3)$$

One useful low-complexity approximation proposed in [7] is obtained by the log-sum approximation. With complete CSI, the approximation leads to

$$\hat{\mathcal{L}}_n^{(i)} = \ln\left(\frac{\max_{x_n \in \mathcal{X}_0(i)} P(y_n|x_n, h_n)}{\max_{x_n \in \mathcal{X}_1(i)} P(y_n|x_n, h_n)}\right), \quad (4)$$

and with full statistical CSI, this leads to

$$\hat{\mathcal{L}}_n^{(i)} = \ln\left(\frac{\max_{x_n \in \mathcal{X}_0(i)} \int_{-\infty}^{\infty} P(y_n|x_n, h)p(h)\, dh}{\max_{x_n \in \mathcal{X}_1(i)} \int_{-\infty}^{\infty} P(y_n|x_n, h)p(h)\, dh}\right). \quad (5)$$

Note that the log-sum approximation is particularly useful under complete CSI assumption since a linear LLR approximation can be implemented [7].

III. A Bayesian Approach for LLR Calculation Using Partial Statistical CSI

Prior works such as [9] provided LLR closed-form expressions considering full statistical CSI over a normalized Rayleigh fading channel for the BSPK, pulse and quadrature amplitude modulations (denoted as PAM and QAM respectively). Then, to address complexity issues, LLR approximations based on Taylor series have been proposed [9]. If the proposed solution provides an interesting framework for the derivation of non linear LLR approximations, it still comes with some limitations. First, this solution considers full statistical CSI (full knowledge of $p(h)$), which is unlikely to be available at the receiver. Moreover, complete derivation of useful expressions is only available for channels for which a convenient analytical expression is available, which is the case for the normalized Rayleigh fading channel scenarios that have been considered, but it will be not that easy to generalize it to any kind of fading channels.

In this letter, we consider a rather different approach considering that only partial statistical CSI (i.e. only first and second order statistics of h) are available at the receiver. Moreover, the proposed method can be implemented independently of the fading channel distribution $p(h)$. Then, whereas σ^2 is assumed to be known or accurately estimated, h_n is considered as an unknown random variable whose first and second statistical moments are well characterized.

The problem of computing LLR values turns out to be the derivation of a closed-form expression of the integral in (3), for which we have to select a suitable prior distribution for the r.v. h, enabling both a good approximation of the true prior distribution and the ease of a closed-form derivation. A common approach in Bayesian analysis, when possible, is to select a prior distribution to be the conjugate of the likelihood distribution, which results in a posterior distribution that is of the same family as the a priori distribution [11]. Given that the likelihood distribution is a Gaussian distribution, the conjugate prior distribution for the r.v. h_n is also a Gaussian one [10], i.e. $h_n \sim \mathcal{N}(\mu_h, \sigma_h^2)$, where the parameters μ_h and σ_h^2 are considered to be known or well estimated. As an example, if the scale factor a of the unnormalized Rayleigh distribution is known, the first and second moments of h can be computed as $\mu_h = a\sqrt{\frac{\pi}{2}}$ and $\sigma_h^2 = \frac{4-\pi}{2}a^2$.

A. M-Ary PAM and QAM Modulations

Based on the selected prior, a closed-form expression for equation (3) can be derived for M-ary PAM modulations, for which $x_n \in \{\pm 1, \pm 3, \cdots \pm (M-1)\}$. The LLR in (3) can be written as

$$\mathcal{L}_n^{(i)} = \ln\left(\frac{\sum_{x_n \in \mathcal{X}_0(i)} \int_{-\infty}^{\infty} e^{-\frac{(y_n - hx_n)^2}{2\sigma^2}} e^{-\frac{(h-\mu_h)^2}{2\sigma_h^2}}\, dh}{\sum_{x_n \in \mathcal{X}_1(i)} \int_{-\infty}^{\infty} e^{-\frac{(y_n - hx_n)^2}{2\sigma^2}} e^{-\frac{(h-\mu_h)^2}{2\sigma_h^2}}\, dh}\right). \quad (6)$$

which can be shown to be (cf. Appendix A):

$$\mathcal{L}_n^{(i)} = \ln\left(\sum_{x_n \in \mathcal{X}_0(i)} e^{-\frac{(y_n - x_n\mu_h)^2}{2\left(\sigma^2 + x_n^2\sigma_h^2\right)}}\right) - \ln\left(\sum_{x_n \in \mathcal{X}_1(i)} e^{-\frac{(y_n - x_n\mu_h)^2}{2\left(\sigma^2 + x_n^2\sigma_h^2\right)}}\right). \quad (7)$$

In order to reduce the complexity, the log-sum approximation can be used following (5). Considering M-ary QAM modulations built as a direct product of two orthogonal Gray encoded PAM constellations, the LLR values can be directly computed by the previous method (i.e. equation (7)) considering independently the two dimensions of the signal. Then, the combined LLR values can be computed as the sum of the LLR values obtained for each of the signal components.

B. BPSK Modulation: A Particular Case

The BPSK modulation can be seen as a particular case of the M-ary PAM modulation where the posterior distribution becomes the product of two Gaussian distributions, and the marginal distribution can be shown to be another Gaussian distribution of the form $p(x_n|y_n) \propto \mathcal{N}(y_n/\mu_h, \left(\sigma^2 + \sigma_h^2\right)/\mu_h^2)$. From the previous distribution, the LLR can be directly computed as

$$\mathcal{L}_n = -\mu_h^2 \frac{\left(1 - \frac{y_n}{\mu_h}\right)^2}{2\left(\sigma^2 + \sigma_h^2\right)} + \mu_h^2 \frac{\left(-1 - \frac{y_n}{\mu_h}\right)^2}{2\left(\sigma^2 + \sigma_h^2\right)} = \frac{2\mu_h}{\left(\sigma^2 + \sigma_h^2\right)} y_n \,. \tag{8}$$

given that $x_n \in \mathcal{X}_0(i) = 1$ and $x_n \in \mathcal{X}_1(i) = -1$. The resulting LLR value is a linear function of y_n.

This result can be linked to previous work by [8], where the authors aim to estimate the linear coefficient $\alpha \in \mathbb{R}^+$ that provides the best linear approximation of the LLR written as $\hat{\mathcal{L}}_n = \alpha y_n$. To this end, [8] proposed to compute the scaling factor α by maximizing an approximate mutual information based quantity, referred to as $\hat{I}\left(\hat{\mathcal{L}}; X\right)$, between the transmitted symbol X and the detector input $\hat{\mathcal{L}}$. The proposed optimization problem can be stated as:

$$\begin{aligned} \alpha &= \arg\max_{\alpha' \in \mathbb{R}^+} \hat{I}\left(\hat{\mathcal{L}}; X\right) \\ &= \arg\max_{\alpha' \in \mathbb{R}^+} 1 - \int_{-\infty}^{\infty} \log_2\left(1 + e^{-\hat{\mathcal{L}}}\right) p\left(\hat{\mathcal{L}}|X = +1\right) d\mathcal{L} \,. \end{aligned} \tag{9}$$

Originally, the optimization method proposed in [8] assumes the knowledge of the linearly approximated LLRs conditional pdf. In some specific cases, as for example the normalized Rayleigh fading channel, an exact analytical expression of LLRs can be derived [8, eq. (17)]. Apart from these specific cases, one has to resort to a numerical optimization method, that can be computationally demanding. It can be done by applying one-dimensional search method [12] based on the objective function of equation (9). To evaluate this integral, as previously stated, one needs the integrand kernel $p\left(\hat{\mathcal{L}}|X = +1\right)$, which is not an easy task to evaluate online. To overcome this difficulty, one can resort to the corresponding empirical mean estimator as done in [13], [14]. But, one still has to resort to iterative one-dimensional search methods with a cost function involving log/exp function evaluations. With the proposed method, we rather need to evaluate both first and second order moments of the random variable h. This shows that for a first order approximation of the LLR, minimizing a functional involving a complete statistical pdf characterization is not a necessary condition to get a good approximation. When full statistical CSI is available, the proposed approach enables to circumvent the above optimization procedure by a direct parametric estimation of the scaling factor α using the first and second order moments, which are easily handled in this case. If full statistical CSI is not available, in order to compute an estimation for the scaling parameter α, we have to estimate online the parameters (μ_h, σ_h^2) from the data at the receiver. Additionally, we can notice that, under an AWGN channel assumption for which $\mu_h = 1$ and $\sigma_h^2 = 0$, the result obtained in equation (8) corresponds to the classical Gaussian LLR expression with $\mathcal{L}_n = 2\, y_n/\sigma^2$.

C. On the Estimation of the Parameters μ_h and σ_h^2

In the previous section, we addressed the issue of computing LLR values considering partial statistical CSI, i.e. the first (μ_h) and the second (σ_h^2) orders of the $p(h)$ are considered known. However, in real scenarios these parameters might not be available at the receiver and should be estimated online. As a simple example, assuming that a binary learning sequence is available at the receiver, and considering the output signal model in (1), i.e. $y_n \sim \mathcal{N}(\mu_h x_n, \sigma_h^2 + \sigma^2)$, we can compute the log-likelihood function $\Lambda(y_n; \mu_h, \sigma_h^2)$ as

$$\log\left(\Lambda\right) = -\sum_{n=1}^{N} \frac{1}{2} \log\left(2\pi\left(\sigma_h^2 + \sigma^2\right)\right) - \sum_{n=1}^{N} \frac{\left(y_n - \mu_h x_n\right)^2}{2\left(\sigma_h^2 + \sigma^2\right)} \,. \tag{10}$$

Then, we can derive maximum likelihood estimates of μ_h and σ_h^2 as the roots of the partial derivatives of (10) with respect to μ_h and σ_h^2.

$$\hat{\mu}_h = \frac{1}{N} \sum_{n=1}^{N} y_n x_n \,, \quad \hat{\sigma}_h^2 = \frac{1}{N} \sum_{n=1}^{N} \left(y_n - \hat{\mu}_h\right)^2 - \sigma^2 \,. \tag{11}$$

Other types of estimation strategies can be also considered, but they are out-of scope of this letter.

IV. Results

In Fig. 1, we compare LLR values obtained as a function of the channel output y_n for a 8-PAM signal set with Gray labeling for a normalized Rayleigh fading channel at $SNR = 7.91$ dB and for the following scenarios : (a) full statistical CSI [9, eq. (9)], (b) their Taylor approximation [9, eqs. (24),(26),(27)]; and (c) the Bayesian approach proposed in (7) (perfect partial statistical CSI). Note that a normalized Rayleigh distribution was used in [9] to compute the LLR values, since a derivable closed-form conditional pdfs is required. From the plots, we note that the proposed Bayesian method exhibits the same behavior as the Taylor approximations for low amplitude values and differs when amplitudes increase. In Fig. 3, we compare Frame Error Rate (FER) between the LLR computed with (a) full statistical CSI [9, eq. (9)], (b) the proposed Bayesian approach, and (c) the Bayesian approach with the log-max approximation. We consider a data frame encoded by an irregular LDPC of rate $1/4$ as defined in the norm DVB-S.2 [15] ($N = 64800$) following the coding rate considered in [9]. For the LDPC decoding, we consider the belief propagation (BP) algorithm [2] with 100 decoding iterations. The proposed Bayesian approach achieves performance with a gap 0.4 dB with respect to the LLR values computed considering full statistical CSI. Note that when the log-sum approximation is used, a gap 0.8 dB is found. We underline that the Bayesian approach only needs partial statistical CSI and not full statistical CSI as in previous works.

In this section we also compare soft decoding performance for a BPSK modulation corresponding to the LLR considering complete CSI (1), the LLR considering full statistical

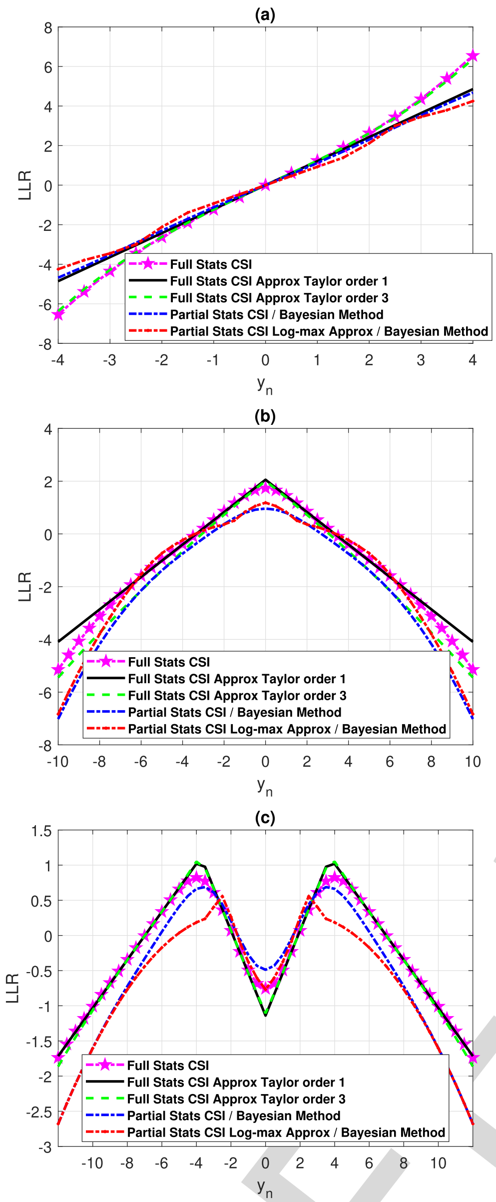

Fig. 1. LLR values $\mathcal{L}_n^{(1)}$ (a), $\mathcal{L}_n^{(2)}$ (b) and $\mathcal{L}_n^{(3)}$ (c) as functions of the channel output y_n for 8-PAM modulation under a normalized Rayleigh channel at $SNR = 7.91$ dB.

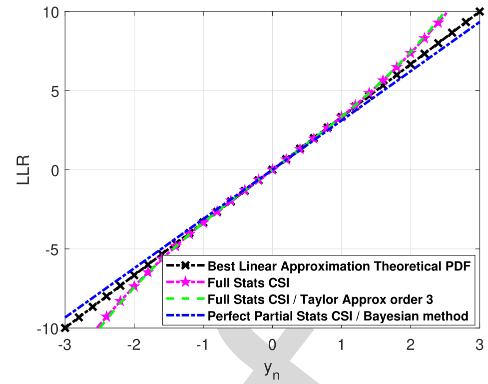

Fig. 2. GPS L1C frame error rate under a Rayleigh channel with $a = 0.2$.

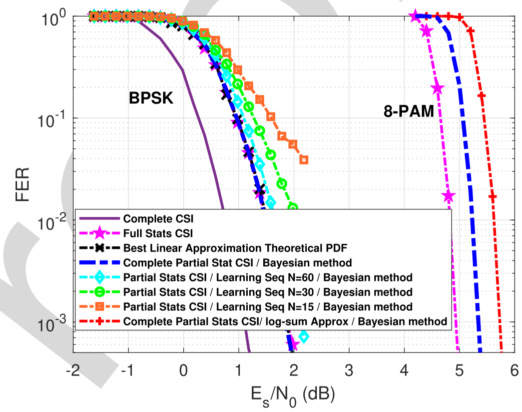

Fig. 3. GPS L1C frame error rate under a normalized Rayleigh channel.

CSI [9, eq. (8)], the best linear approximation of the LLR proposed in [8] (full statistical CSI) and the Bayesian approach to compute LLR (8) considering partial statistical CSI. In particular, as an example, we provide (FER) performance for the GPS L1C subframe 2 [16], [17] ($N = 1200$), which is based on an irregular LDPC code of rate $1/2$ and decoded by the BP algorithm. We consider a normalized Rayleigh fading channel (since an analytical expression of the LLRs pdf is necessary to compute both: the LLR expression in [9, eq. (8)] and the best linear approximation method). In Fig. 2, we plot the LLRs as a function of the observation y_n at $E_b/N_0 = 4.5$ dB. Note from Fig. 2 that the Bayesian approach (8) (when partial statistical CSI is assumed) converges to the same LLR values than the best linear approximation approach, whereas the proposed method does not involves full statistical CSI. Moreover, considering the same fading channel distribution and the same methods to compute LLRs, the FER for the previous methods exhibit again similar behaviors as shown in Fig. 3. This illustrates the fact that the proposed method does not suffer from any loss of information compared to other methods. When μ_h and σ_h^2 are estimated from a learning sequence of length $N_p = 60$ symbols, a small degradation of 0.2dB for the FER is observed. Note that for the particular case of GPS, a pilot component is transmitted in parallel to the data component, a larger N_p could be considered to estimate μ_h and σ_h^2). This degradation increases when the number of symbols to estimate μ_h and σ_h^2 is reduced. The method used to estimate μ_h and σ_h^2 is provided in subsection III-C. Note that the method in [9] (not reported in the figure) has similar performance to the full statistical CSI method.

Finally, we consider the case of an unnormalized Rayleigh channel with a scale factor of $a = 0.2$. Fig. 4 shows the corresponding FER performance. Note that, for this experiment, no analytical expression for the LLRs pdf is available. Therefore, the LLR considering full statistical CSI [9, eq. (8)] cannot be computed. In order to compute the best linear approximation method, since no analytical expression for the LLRs pdf is available, the empirical estimator proposed in [13] is used in order to provide an estimation of the coefficient α, defined in (8). Similar conclusions to the previous case can be drawn for the FER performance. We underline, that thanks to the Bayesian approach, full statistical CSI is not required. Then, the complexity of the method consist on estimating μ_h and σ_h^2, i.e. to compute (10).

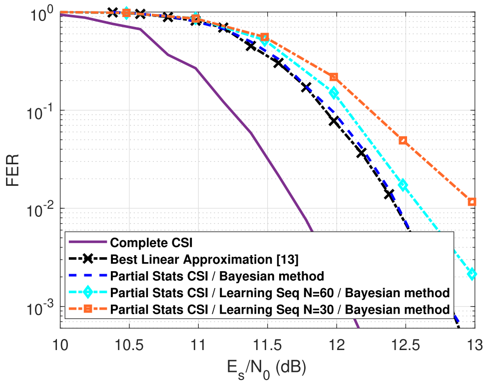

Fig. 4. GPS L1C frame error rate under a Rayleigh channel with $a = 0.2$.

V. Conclusion

In this letter, we have addressed the problem of the derivation of LLR values approximations for uncorrelated fading channels using partial statistical CSI. To this end, we have proposed a different method following a Bayesian approach using conjugate prior analysis. Under this framework, we are able to derive a simple closed-form solution of the conditional pdf. Then, we can obtain an analytical closed-form expression of the LLR values, which are independent of the fading gain distribution $p(h)$. Moreover, this solution can be shown to be only dependent on the first and second order moments associated with the random variable h. As a consequence, based on this analysis, it appeared that full statistical CSI is not a sole condition to derive accurate LLR functions, but partial statistical CSI based on statistics of order 1 and 2 can also lead to accurate and robust approximations. Finally, we have presented a simple method to compute online estimation of statistics of order 1 and 2 when a learning sequence is available at the receiver, showing that the proposed method can be implemented with a reasonable complexity.

Appendix A

In this appendix, we solve the integral in (6):

$$\begin{aligned} p(x_n|y_n) &\propto \int_{-\infty}^{\infty} e^{-\frac{(y_n - hx_n)^2}{2\sigma^2}} e^{-\frac{(h-\mu_h)^2}{2\sigma_h^2}} dh \\ &= \int_{-\infty}^{\infty} e^{-\beta_1\left(y_n^2 - 2hx_ny_n + h^2{x_n}^2\right)} e^{-\beta_2\left(h^2 - 2h\mu_h + \mu_h^2\right)} dh, \end{aligned} \tag{12}$$

where $\beta_1 = \frac{\sigma_h^2}{2\sigma^2\sigma_h^2}$ and $\beta_2 = \frac{\sigma^2}{2\sigma^2\sigma_h^2}$. Since the product of two Gaussian distributions is a Gaussian distribution, we proceed by finding the mean μ_a and variance σ_a^2 of the resulting Gaussian distribution as

$$\begin{aligned} \frac{(h-\mu_a)^2}{\sigma_a^2} + \kappa = \beta_1\left(y_n^2 - 2hx_ny_n + h^2{x_n}^2\right) \\ + \beta_2\left(h^2 - 2h\mu_h + \mu_h^2\right) \end{aligned} \tag{13}$$

where κ is an auxiliary constant. Expanding the expressions

$$\begin{aligned} &\frac{h^2}{\sigma_a^2} - \frac{2h\mu_a}{\sigma_a^2} + \frac{\mu_a^2}{\sigma_a^2} + \kappa \\ &= \beta_1 y_n^2 + \beta_2\mu_h^2 - 2h\left(x_ny_n\beta_1 + \mu_h\beta_2\right) + h_n^2\left(\beta_1{x_n}^2 + \beta_2\right), \end{aligned} \tag{14}$$

it follows that $\frac{1}{\sigma_a^2} = \left(\beta_1{x_n}^2 + \beta_2\right)$, $\frac{\mu_a}{\sigma_a^2} = \left(x_ny_n\beta_1 + \mu_h\beta_2\right)$ and $\frac{\mu_a^2}{\sigma_a^2} = \frac{(x_ny_n\beta_1+\mu_h\beta_2)^2}{(\beta_1{x_n}^2+\beta_2)}$ after identifying terms on both sides of equation (14). The constant κ can be computed as

$$\kappa = \frac{\beta_1\beta_2}{\beta_1{x_n}^2 + \beta_2}\left(y_n^2 - 2x_ny_n\mu_h + \mu_h^2{x_n}^2\right) = \frac{\left(y_n - x_n\mu_h\right)^2}{2\left(\sigma^2 + {x_n}^2\sigma_h^2\right)},$$

where $\frac{\beta_1\beta_2}{\beta_1{x_n}^2+\beta_2} = \frac{1}{2\left(\sigma^2+{x_n}^2\sigma_h^2\right)}$. Reporting these equations, equation (12) can be re-written as

$$\int_{-\infty}^{\infty} e^{-\frac{(h-\mu_a)^2}{\sigma_a^2}} e^{-\frac{(y_n - x_n\mu_h)^2}{2\left(\sigma^2+{x_n}^2\sigma_h^2\right)}} dh = e^{-\frac{(y_n - x_n\mu_h)^2}{2\left(\sigma^2+{x_n}^2\sigma_h^2\right)}} \tag{15}$$

where by definition we have $\int_{-\infty}^{\infty} e^{-\frac{(h-\mu_a)^2}{\sigma_a^2}} dh = 1$, yielding to equation (7) after inserting equation (15) into equation (6).

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