Lie group based approach for GNSS Signal Phase modeling

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***Abstract*—Leveraging carrier phase observations within Global Navigation Satellite Systems receivers allows centimeter-level positioning accuracy. However, carrier phase observations are significantly affected by additive noise, which is assumed to follow a von Mises distribution, thereby degrading the performance of phase-based positioning estimators. To improve the modeling of carrier phase observations, we propose a novel approach that constrains the parameters of the von Mises distribution—specifically, the angular location modeling the phase and its dispersion parameter κ modeling the noise—to evolve within the Lie group space $SO(2) \times \mathbb{R}^+$. To estimate these parameters, we employ a Lie group maximum likelihood estimator, solved through a Newton algorithm on Lie groups. This approach demonstrates advantages in terms of robustness and precision, especially when dealing with a small number of observations, compared to traditional Euclidean-based methods.**

***Index Terms*—GNSS, Lie group, phase, von Mises distribution, concentration parameter, estimation.**

I. Introduction

Intelligent transportation systems and safety-critical applications are gaining prominence in today's society. They require precise and reliable positioning services to function effectively in complex environments. Global Navigation Satellite Systems (GNSS) is the technology of choice when it comes to high precision navigation, enabling centimeter-level positioning accuracy. This requires the use of carrier phase measurement (see [1]) and in particular to solve for the integer carrier phase ambiguities [2], an operation sensitive to low Signal-to-Noise Ratio (SNR), which can significantly affect the precision of receiver position estimates. Although it can be tedious, the process of ambiguity fixing opened the door for the Real-Time Kinematic (RTK) and the Precise Point Positioning (PPP) techniques for high-precision positioning [3]. As opposed to the RTK technique, the PPP one is powerful because it does not require for the user to rely on a dense station network and to stay in the vicinity of a reference station to reach a high positioning accuracy. Hence it has been widely used from several decades up to recent applications, either public or commercial [4], [5], [6], [7], [8].

In this study, we investigate the modeling of the carrier phase within the tracking loop of a GNSS receiver using a statistical approach designed to be resilient to noise. This approach represents an initial step towards achieving robust and accurate position estimation in challenging environments, as well as enabling high-precision positioning and navigation.

Conventional GNSS tracking loops implement a phase lock loop (PLL) to estimate both the carrier phase and the Doppler frequency variations over time. The tracking stage can be seen as a recursive estimation problem, and the PLL architecture can be reformulated as a Kalman-like filter [9]. Traditional sequential estimation filters, which rely on Gaussian assumptions, fail to account for the periodicity of circular data and thus produce biased results [10], [11], since they are not defined in a circular domain within the interval $]-\pi, \pi]$. In [12], the authors have shown that the phase noise of a GNSS signal follows a von Mises distribution (VMD), [3], [13], and such a modelisation was extensively used for angle modeling within different contexts [14], [15], [16]. It is characterized by two unknown parameters which are the concentration parameter, κ, and the location parameter, ϕ corresponding to the phase effectively to track in the PLL. In an operational context, the phase variance of a signal is not well-known due to the lack of precise information about the Signal-to-Noise Ratio (SNR). In the GNSS context, the quality of the signal is commonly assessed by estimating the Carrier-to-Noise Ratio (C/N_0), which provides critical information about the health of the satellite signal [17]. Additionally, the concentration parameter κ, which quantifies the dispersion of phase measurements, is directly related to phase noise. This relationship links κ to the quality of the signal, as C/N_0 and κ are inherently connected [18], [15]. Consequently, in this work, we aim to model phase measurements using a VMD to estimate not only the phase angle ϕ but also the concentration parameter κ, which provides valuable insights into the signal quality.

Deriving an estimator for κ is challenging since it does not admit analytical expressions. In the state-of-the art, different

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approaches exist. Iterative robust algorithms for κ and ϕ adapted from the Maximum Likelihood Estimator (MLE) were proposed in [19]. In [20], a maximum likelihood (ML) approach for estimating the concentration parameter is proposed, but it results in a highly biased performance dependent on the number of measurements. In contrast, other estimators discussed in [21] reduce the bias but lack of precision for small κ values.

To overcome these defaults, we propose to model the VMD with its parameters constrained to live on their natural space $]-\pi,\pi]$ and $\mathbb{R}^+$. This can be naturally accomplished by applying their Lie group structure. A Lie group is a differentiable manifold equipped with a structure of group, in which an Euclidean motion can be described as a rotation in the Lie group space. They are classically used in other fields such as machine learning and signal processing, to model physical quantities as rotation matrices $SO(n)$ and affine transformations $SE(n)$ [22]. Then, ϕ is associated to a rotation matrix $\mathbf{R}$ in $SO(2)$ and κ lies on the Lie group $\mathbb{R}^+$. Knowing the latter, we derive a ML estimator of ($\mathbf{R}$, κ) on Lie group for VMD, resolved by a Newton algorithm on the Lie group $SO(2)\times\mathbb{R}^+$. In this way, we ensure that, at each iteration of the algorithm, ϕ and κ are both estimated on $]-\pi,\pi]$ and $\mathbb{R}^+$, naturally and without additional constraints. Compared to conventional methods, this approach demonstrates robustness in adverse scenarios characterized by small values of κ (indicating high variance) and a limited number of measurements, which are common in GNSS receivers. This paper is organized in the following way: section II details the conventional approach to estimate the VMD parameters. Section III briefly provides the background on Lie groups and introduce the proposed approach. Then, numerical simulations are presented in section IV. Finally, the conclusion and future perspectives are discussed in section V.

II. Classical Euclidean Approach

In this section, we provide the necessary background to characterize the VMD. First, we introduce its definition and explain its relation with the GNSS application. Second, we detail the classical technique to estimate its parameters, i.e. the location parameter, ϕ, and the concentration parameter, κ.

A. Definition

The VMD is classically encountered in signal processing to characterize angle or phase measurements, and it has the advantage to have a support on $]-\pi,\pi]$ [11]. Therefore, considering the phase measurement as a random variable following a VMD $\psi\sim\mathcal{VM}(\phi,\kappa)$, its probability density function is given by:

$$f(\psi|\phi,\kappa)=\frac{1}{2\pi\mathrm{I}_0(\kappa)}\exp\left\{\kappa\cos(\psi-\phi)\right\}, \tag{1}$$

where $\mathrm{I}_0(.)$ is the 0 order modified Bessel function.

B. Von Mises distribution in GNSS signal model

In the context of GNSS, the received signal, z, at each discrete instant i, can be modeled by:

$$z_i=\alpha e^{j\phi_i}+n_i, \tag{2}$$

where α is the signal amplitude, n_i a white complex Gaussian noise with variance σ_n^2 defined as $n_i\sim\mathcal{CN}\left(0,\sigma_n^2\right)$, and ϕ the true signal phase. The phase measure ψ in the typical GNSS measurement context can be defined as $\psi_i = \arctan(\mathrm{Im}\{z_i\},\mathrm{Re}\{z_i\}) \in]-\pi,\pi]$. According to this modeling, it can be demonstrated that the signal phase measurements ψ_i follows a VMD with location parameter ϕ and concentration parameter $\kappa=2\alpha/\sigma_n^2$ [15].

C. Conventional VMD parameter estimation

Within the tracking loops of a GNSS receiver, the problem of estimating and denoising the phase information, ϕ, from N independent von Mises observations $\{\psi_i\}_{i=1}^N$, and subsequently computing its variance, is fundamental. A conventional approach to this problem consists in identifying the pair of parameters, such as $\mathbf{x}=[\phi,\kappa]^\top$, that maximize the VM likelihood, as follows:

$$\hat{\mathbf{x}}=\underset{\phi,\kappa}{\operatorname{argmax}}\prod_{i=1}^N\frac{1}{2\pi\mathrm{I}_0(\kappa)}\exp\left\{\kappa\cos(\psi_i-\phi)\right\}. \tag{3}$$

Equation (3) can be reformulated in order to minimize the negative logarithm, thereby defining the estimation problem as the following optimization problem as follows:

$$\hat{\mathbf{x}}=\underset{\phi,\kappa}{\operatorname{argmin}}\,h(\phi,\kappa) \tag{4}$$

where:

$$h(\phi,\kappa)=\sum_{i=1}^N\left[\log\left(2\pi\mathrm{I}_0(\kappa)\right)-\kappa\cos(\psi_i-\phi)\right]. \tag{5}$$

This problem does not admit any analytical solution due to the presence of the cosine function. Indeed, the latter has cyclic local minima. In order to solve this optimization problem, a numerical solution has to be adopted. While gradient descent and other fixed-point algorithms have been applied in previous work [16], the Newton algorithm is also classically used for its low complexity and its fast convergence properties [23]. The recursion is basically given, at each iteration l, by the optimization of the criteria in equation (5) as follows:

$$\hat{\mathbf{x}}_{l+1}=\hat{\mathbf{x}}_l-\left(\frac{\partial^2}{\partial^2\mathbf{x}}h(\hat{\mathbf{x}}_l)\right)^{-1}\frac{\partial}{\partial\mathbf{x}}h(\hat{\mathbf{x}}_l). \tag{6}$$

Note that the resulting estimator of κ is highly biased and that there exists various methods to compensate for it [21], [24].

D. Limitations of conventional approach

Equation (6) shows that the subtraction operator in the Newton algorithm can cause the estimated parameters to fall outside their natural ranges at any given instant. To address this issue, one solution would be to constrain the estimate of κ to lie within $\mathbb{R}^+$ by using its absolute value. As for ϕ, applying the modulus operation to its estimated value will ensure it remains within its specified range. However, these empirical solutions are theoretically inadequate and are not mathematically rigorous from a statistical perspective since this type of adjustment modifies the uncertainty of the model, adding new challenges for future work based on recursive filtering where the uncertainty of the model must be defined.

III. Proposed strategy on Lie groups

Due to the previously mentioned problem, we propose an approach to estimate the VMD parameters within the Lie group (LG) framework. By using rotation matrices on $SO(2)$ we can dispense with the modulo constraint associated with phase angles. Then, we reformulate the VMD as a function of a rotation matrix lying on $SO(2)$. By using the LG structure of the concentration parameter on $\mathbb{R}^+$, we deduce a LG estimation problem that we propose to resolve with a Newton algorithm on LG, requiring the computations of gradient and hessian on LGs.

A. Lie group (LG): definition and examples

1) Definition: A matrix Lie group $(G \subset \mathbb{R}^{n\times n}, \circledast)$ is a set of $n \times n$ matrices that form a smooth manifold and a group under the operation $\circledast$. The tangent space at the identity matrix, called the Lie algebra $\mathfrak{g}$, is a vector space that can be seen as a local approximation of G. Elements of the $\mathfrak{g}$ are linked to the LG through exponential and logarithm maps $\mathrm{Exp}_G : \mathfrak{g} \to G$ and $\mathrm{Log}_G : G \to \mathfrak{g}$. If $\mathfrak{g}$ is with dimension m, we can use bijections and its reciprocal $[.]^\wedge : \mathbb{R}^m \to \mathfrak{g}$ and $[.]^\vee : \mathfrak{g} \to \mathbb{R}^m$ to move between $\mathfrak{g}$ and $\mathbb{R}^m$. This structure is advantageous because it allows to deal with Euclidean vectors rather than with matrices which is more suitable in a numerical point of view [25], [23], [26]. In Fig. 1 a schematic with the relation between the Lie group, Lie algebra and Euclidean space is shown.

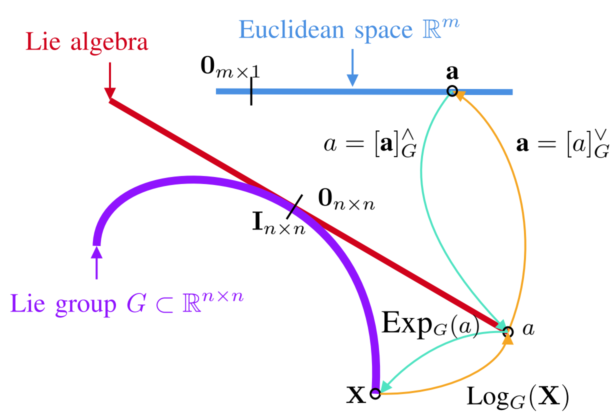

Fig. 1: Relation between Lie group, Lie algebra and Euclidean space.

2) Case of $SO(n)$: $SO(n)$ represents the set of the rotation matrices verifying the following property $SO(n) = \left\{\mathbf{R} \in \mathbb{R}^{n\times n}, \mathbf{R}\mathbf{R}^T = \mathbf{I}\right\}$. In the case where $n = 2$, $\mathbf{R}$ is parameterized with respect to ϕ such as:

$$\mathbf{R} = \begin{bmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{bmatrix}, \forall \phi \in \mathbb{R}, \tag{7}$$

its Lie algebra is given by the set of anti-symmetric matrices:

$$\mathbf{T} = \begin{bmatrix} 0 & -\phi \\ \phi & 0 \end{bmatrix}, \forall \phi \in \mathbb{R}, \tag{8}$$

and the exponential map of $SO(2)$ is given by matrix exponential as

$$\mathrm{Exp}^\wedge_{SO(2)}(\phi) = \exp\left\{\mathbf{T}\right\}. \tag{9}$$

Regarding the logarithm application, it is naturally given by the matrix logarithm

$$\mathrm{Log}_{SO(2)}(\mathbf{R}) = \begin{bmatrix} 0 & -\phi \\ \phi & 0 \end{bmatrix},$$

we observe that the associated Euclidean space is with dimension 1 generated by ϕ then $\mathrm{Log}^\vee_{SO(2)}(\mathbf{R}) = \phi$.

3) Case of $\mathbb{R}^+$: The space $\mathbb{R}^+$ forms a commutative LG under the operation of classical multiplication, with the neutral element being 1. The logarithm and exponential maps for this group are given by the natural logarithm $\log(.)$ and the exponential function $\exp(.)$, respectively, both defined on $\mathbb{R}^+$ and $\mathbb{R}$.

B. VMD estimation problem

In this subsection, we propose a new modeling of the VMD. As evoked previously, an angle ϕ can be parameterized by a rotation matrix on $SO(2)$. Then, the angular parameter ϕ of the VMD defined by equation (1) can be written as $\phi = \mathrm{Log}^\vee_{SO(2)}(\mathbf{R})$ with $\mathbf{R} \in SO(2)$, and we obtain:

$$f(\psi_i|\mathbf{R},\kappa) = \frac{1}{2\pi \mathrm{I}_0(\kappa)} \exp\left\{\kappa \cos[\psi_i - \mathrm{Log}^\vee_{SO(2)}(\mathbf{R})]\right\}. \tag{10}$$

Furthermore, it is worth noting that, κ being positive, it belongs to the LG of the positive value. Consequently, the set of the unknown parameters belongs to the LG product $SO(2)\times\mathbb{R}^+$ and is written bellow in equation (11). To estimate them, we propose to find the following ML estimator on LGs:

$$\mathbf{X} = \begin{bmatrix} \mathbf{R} & \mathbf{0}_{2\times 1} \\ \mathbf{0}_{1\times 2} & \kappa \end{bmatrix}, \tag{11}$$

$$\widehat{\mathbf{X}} = \underset{\mathbf{R},\kappa}{\mathrm{argmax}} \prod_{i=1}^{N} \frac{1}{2\pi \mathrm{I}_0(\kappa)} \exp\left\{\kappa \cos(\psi_i - \mathrm{Log}^\vee_{SO(2)}(\mathbf{R}))\right\}. \tag{12}$$

By taking the opposite logarithm of the criterion to maximize in (12), we yield:

$$\widehat{\mathbf{X}} = \underset{\mathbf{R},\kappa}{\mathrm{argmin}}\, h(\mathbf{R},\kappa), \tag{13}$$

where:

$$h(\mathbf{R},\kappa) = \sum_{i=1}^{N} \left[\log\left(2\pi \mathrm{I}_0(\kappa)\right) - \kappa \cos(\psi_i - \mathrm{Log}^\vee_{SO(2)}(\mathbf{R}))\right]. \tag{14}$$

C. Resolution of the estimation problem: Newton algorithms on LGs

Similarly to the Euclidean case, the estimation problem in equation (14) is not resolvable analytically but rather by a numerical approach based on a Newton algorithm on LGs. It has been intensively used in the literature to find a local minima of a non-convex quadratic problem on LGs [23], [26].

More precisely, $\widehat{\mathbf{X}}$ can be approached by generating a set of $\{\mathbf{X}^l\}_{l=1}^L$ converging towards $\mathbf{X}$:

$$\mathbf{X}^{l+1} = \mathbf{X}^l \mathrm{Exp}^{\wedge}_{SO(2)\times\mathbb{R}^+}[\boldsymbol{\delta}^l], \quad \forall\, l \in [\![1:L]\!],$$
$$\text{with } \mathrm{Exp}^{\wedge}_{SO(2)\times\mathbb{R}^+}[\boldsymbol{\delta}^l] = \begin{bmatrix} \mathrm{Exp}^{\wedge}_{SO(2)}(\delta^l_{\mathbf{R}}) & 0 \\ 0 & \exp(\delta^l_\kappa) \end{bmatrix}, \tag{15}$$

and $\boldsymbol{\delta}^l = [\delta^l_{\mathbf{R}}, \delta^l_\kappa]^\top$ (with $\delta^l_{\mathbf{R}}$ and $\delta^l_\kappa \in \mathbb{R}$) is:

$$\begin{bmatrix} \delta^l_{\mathbf{R}} \\ \delta^l_\kappa \end{bmatrix} = -\mathbf{H}(\mathbf{R}^l,\kappa^l)^{-1}\boldsymbol{\nabla} h(\mathbf{R}^l,\kappa^l), \tag{16}$$

where $\mathbf{H}(\mathbf{R},\kappa) \in \mathbb{R}^{2\times 2}$ and $\boldsymbol{\nabla} h(\mathbf{R},\kappa) \in \mathbb{R}^2$ are the Hessian matrix and gradient of $h(\mathbf{R},\kappa)$ respectively. Since both operations are defined in $SO(2)\times\mathbb{R}^+$ space, hence the LG Hessian and gradient are defined as follows [27]:

$$\mathbf{H}(\mathbf{R},\kappa) = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}, \tag{17}$$

where the components of $\mathbf{H}$ belong to $\mathbb{R}$ and are defined as:

$$H_{11} = \left.\frac{\partial h(\mathbf{R}\mathrm{Exp}^{\wedge}_{SO(2)}(\epsilon'_{\mathbf{R}})\mathrm{Exp}^{\wedge}_{SO(2)}(\epsilon''_{\mathbf{R}}),\kappa)}{\partial\epsilon'_{\mathbf{R}}\partial\epsilon''_{\mathbf{R}}}\right|_{\boldsymbol{\epsilon_H}=0},$$
$$H_{12} = \left.\frac{\partial h(\mathbf{R}\mathrm{Exp}^{\wedge}_{SO(2)}(\epsilon_{\mathbf{R}}),\kappa\exp(\epsilon_\kappa))}{\partial\epsilon_{\mathbf{R}}\partial\epsilon_\kappa}\right|_{\boldsymbol{\epsilon_H}=0},$$
$$H_{21} = \left.\frac{\partial h(\mathbf{R}\mathrm{Exp}^{\wedge}_{SO(2)}(\epsilon_{\mathbf{R}}),\kappa\exp(\epsilon_\kappa))}{\partial\epsilon_\kappa\partial\epsilon_{\mathbf{R}}}\right|_{\boldsymbol{\epsilon_H}=0},$$
$$H_{22} = \left.\frac{\partial h(\mathbf{R},\kappa\exp(\epsilon'_\kappa)\exp(\epsilon''_\kappa))}{\partial\epsilon'_\kappa\partial\epsilon''_\kappa}\right|_{\boldsymbol{\epsilon_H}=0}.$$

The gradient is given by:

$$\boldsymbol{\nabla} h(\mathbf{R},\kappa) = \left.\begin{bmatrix} \frac{\partial h(\mathbf{R}\mathrm{Exp}^{\wedge}_{SO(2)}(\epsilon_{\mathbf{R}}),\kappa)}{\partial\epsilon_{\mathbf{R}}} \\ \frac{\partial h(\mathbf{R},\kappa\exp(\epsilon_\kappa))}{\partial\epsilon_\kappa} \end{bmatrix}\right|_{\boldsymbol{\epsilon_h}=0}, \tag{18}$$

where $\boldsymbol{\epsilon_H} = [\epsilon_\kappa, \epsilon'_\kappa, \epsilon''_\kappa, \epsilon_{\mathbf{R}}, \epsilon'_{\mathbf{R}}, \epsilon''_{\mathbf{R}}]^\top$ and $\boldsymbol{\epsilon_h} = [\epsilon_{\mathbf{R}}, \epsilon_\kappa]^\top$. By using (14), and knowing that

$$\left.\frac{\partial}{\partial\epsilon}\mathrm{Log}^{\vee}_{SO(2)}(\mathbf{R}\,\mathrm{Exp}^{\wedge}_{SO(2)}(\epsilon))\right|_{\epsilon=0} = 1,$$

and $\left.\frac{\partial}{\partial\epsilon_\kappa}\kappa\exp(\epsilon_\kappa)\right|_{\epsilon_\kappa=0} = \kappa$, we can demonstrate that

$$\boldsymbol{\nabla} h(\mathbf{R},\kappa) = \begin{bmatrix} -\sum_{i=1}^{N}\kappa\sin(\psi_i - \mathrm{Log}^{\vee}_{SO(2)}(\mathbf{R})) \\ N\left(\kappa\frac{\mathrm{I}_1(\kappa)}{\mathrm{I}_0(\kappa)}\right) - \sum_{i=1}^{N}\kappa\cos(\psi_i - \mathrm{Log}^{\vee}_{SO(2)}(\mathbf{R})) \end{bmatrix}, \tag{19}$$

$$H_{11} = \sum_{i=1}^{N}\kappa\cos(\psi_i - \mathrm{Log}^{\vee}_{SO(2)}(\mathbf{R})), \tag{20}$$

$$H_{21} = -\sum_{i=1}^{N}\kappa\sin(\psi_i - \mathrm{Log}^{\vee}_{SO(2)}(\mathbf{R})), \tag{21}$$

$$H_{12} = H_{21} \tag{22}$$

$$H_{22} = -\sum_{i=1}^{N}\kappa\cos(\psi_i - \mathrm{Log}^{\vee}_{SO(2)}(\mathbf{R})) + N\left[\frac{\kappa^2}{\mathrm{I}_0(\kappa)}\left[\frac{\mathrm{I}_0(\kappa)+\mathrm{I}_2(\kappa)}{2}\right] - \kappa^2\frac{\mathrm{I}_1(\kappa)^2}{\mathrm{I}_0(\kappa)^2} + \kappa\frac{\mathrm{I}_1(\kappa)}{\mathrm{I}_0(\kappa)}\right], \tag{23}$$

where $\mathrm{I}_0(\kappa)$, $\mathrm{I}_1(\kappa)$ and $\mathrm{I}_2(\kappa)$ are the modified Bessel function of zero, first and second order respectively, with respect to κ.

IV. Numerical Simulation

In this section, we propose to assess numerically the proposed LG modeling by implementing the Newton algorithm estimating $\mathbf{R}$ and κ. To achieve this, we generate N synthetic von Mises phase measurements at different GNSS acquisition times, based on a true phase assumed to be constant over the acquisition interval. The performance of the ϕ estimators is shown in sub-section IV-A for different angles, and for κ estimators in sub-section IV-B. To assess the performance of the estimators, we ran the algorithm on $Nr = 10000$ realizations until convergence. Then, we compute the intrinsic MSE of ϕ and κ by using the difference operation in LG [25] given by:

$$MSE_\phi = \frac{1}{Nr}\sum_{m=1}^{Nr}\left[\mathrm{Log}^{\vee}_{SO(2)}(\mathbf{R}^{-1}\widehat{\mathbf{R}}_m)\right]^2,$$
$$MSE_\kappa = \frac{1}{Nr}\sum_{m=1}^{Nr}\left[\log(\kappa^{-1}\hat{\kappa}_m)\right]^2, \tag{24}$$

where $\widehat{\mathbf{R}}_m$, $\hat{\kappa}_m$ denote the estimator for the m-th run, and $\mathbf{R}$ is the rotation matrix associated to the phase angle to estimate given in equation (7). The advantage of these two metrics, besides calculating the error intrinsically, is that it allows for a more accurate evaluation of the local error variation.

A. Scenario 1 : estimation of ϕ

First, we consider a scenario where κ is known and focus solely on estimating ϕ. In Fig. 2, we plot the MSE_ϕ obtained from $\mathbf{R}$ at the last iteration of the Newton algorithm for different number of observations, where $\hat{\phi}_{EU}$ refers to the Euclidean MLE obtained from equation (6) and $\hat{\phi}_{LG}$ the estimator obtained from equation (15). It is superimposed on the MSE_ϕ obtained with the conventional Newton algorithm. We can observe that both methods provide the same performance and validate the proposed LG modeling. It is also important to mention that as ϕ is close to the value π, the quantity of measurement to reach convergence will increase since the mean of the distribution is shifted to the boundaries of the domain of $]-\pi,\pi]$, thus losing its symmetrical appearance.

B. Scenario 2 : estimation of κ

To observe the benefits of LG modeling, we focus on estimating the parameter κ while knowing ϕ. The κ parameter gives information about the dispersion of the measurements in the von Mises distribution (the higher the κ, the less dispersed they will be and the closer to a Gaussian distribution), which is directly related to the variance of the measurements, and in turn to the C/N_0 as it was mentioned before [18], [15]. In addition to using the metrics defined in (24) and the Euclidean MSE to evaluate performance, we define $\hat{\kappa}_{LG}$ as the proposed LG estimator derived from equation (12), and $\hat{\kappa}_{EU}$ as the Euclidean estimator given in equation (6), to then compare them with state-of-the-art estimators detailed in [20] and [21], which are defined as follows:

- $\hat{\kappa}_{V1}$: bias mitigation estimator applied on $\hat{\kappa}_{EU}$ and given by:

$$\hat{\kappa}_{V1} = \begin{cases} \max\{\hat{\kappa}_{EU} - \frac{2}{N\hat{\kappa}_{EU}}, 0\} & \text{for } \hat{\kappa}_{EU} < 2 \\ \frac{(N-1)^3}{N^3+N}\hat{\kappa}_{EU} & \text{for } \hat{\kappa}_{EU} \geq 2 \end{cases}. \quad (25)$$

- $\hat{\kappa}_{V2}$: another bias mitigation estimator applied to $\hat{\kappa}_{EU}$ and given by:

$$\hat{\kappa}_{V2} = \max\left\{N\hat{\kappa}_{EU} - \frac{\hat{\kappa}_{EU} - 1}{N}\sum_{n=1}^{N}\hat{\kappa}_{EU,n-1}, 0\right\}, \quad (26)$$

  where $\hat{\kappa}_{EU,n-1}$ is the estimator (6). constructed using all measurements except the $(n-1)^{\text{th}}$.
- $\hat{\kappa}_{MED}$: estimator based on a median computation as follows:

$$\hat{\kappa}_{MED} = \frac{0.67724}{\textit{median}\left(\{2\left(1-\cos(\psi_i - \phi)\right)\}_{i=1}^{N}\right)}, \quad (27)$$

  where *median* defines the empirical median of a set of data.

In Figs. 3-6, the intrinsic MSE of each estimator is shown, based on the number of considered measurements in the simulations and for different values of κ. We can mainly denote that $\hat{\kappa}_{LG}$ outperforms the Euclidean estimators for all κ values estimated. Particularly, we observe a much better performance for a small number of measurement ($1 \leq N \leq 4$) in comparison to the other estimators, however $\hat{\kappa}_{V1}$ and $\hat{\kappa}_{EU}$ reach to converge to $\hat{\kappa}_{LG}$ after certain number of measurements. This behavior is particularly relevant in the context of GNSS, where only a single phase measurement may be available at a time.

Regarding Euclidean estimators $\hat{\kappa}_{MED}$ presents a lower error for a number of measurements $N \leq 2$ and small κ values (0.01 to 1), even though it is highly biased, therefore its overall performance is far from reaching the $\hat{\kappa}_{LG}$. The estimator $\hat{\kappa}_{V1}$ shows a more stable behavior than the aforementioned, converging to the LG error as the number of measurements increases (see Figs. 3, 4 and 6), even though it is not defined for $N = 1$, therefore, $\hat{\kappa}_{LG}$ presents an operational advantage.

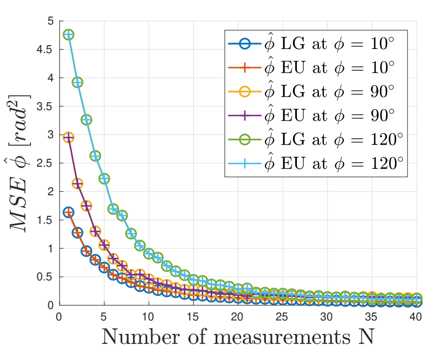

Fig. 2: Intrinsic MSE of $\hat{\phi}$ at a known $\kappa = 1$ for different values of ϕ.

The $\hat{\kappa}_{EU}$ estimator is the estimator with the most stable behavior of all the Euclidean estimators, which, globally, is able to converge the results of $\hat{\kappa}_{LG}$ after several measurements, even though its performance for a limited number of measurements ($1 \leq N \leq 4$) is inferior to that of $\hat{\kappa}_{LG}$. Finally $\hat{\kappa}_{V2}$ was the estimator with the highest error and far from the expected performance because the method used eliminates one available measurement in an estimator highly sensitive to the number of available measurements, this can be observed in the high error values for the Euclidean estimators for a low number of measurements compared to $\hat{\kappa}_{LG}$.

Another interesting behavior is that the larger the κ to estimate, the lower the MSE of each estimator will be, and a large number of measurements are needed to converge. This behavior can be explained by a strong contribution of the bias on the MSE, when the value of κ is high, which consequently becomes more challenging to mitigate. Regarding $\hat{\kappa}_{LG}$, this behavior is not observable, indicating that its bias value (LG-bias) is much lower. For $\kappa = 1$, $\hat{\kappa}_{LG}$ and $\hat{\kappa}_{EU}$ took 79 and 62 seconds respectively, indicating that the LG method is not particularly computationally expensive.Finally, $\hat{\kappa}_{LG}$ appears to be a better estimator than the others tested for a small number of measurements and high κ values, as it is highly impacted by biases when κ decreases.

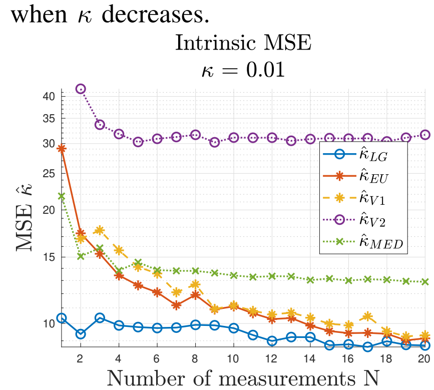

Fig. 3: Intrinsic MSE of each estimator for $\kappa = 0.01$

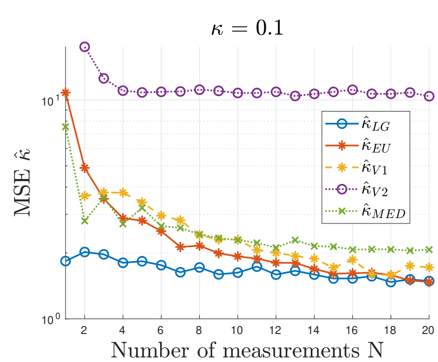

Fig. 4: Intrinsic MSE of each estimator for $\kappa = 0.1$

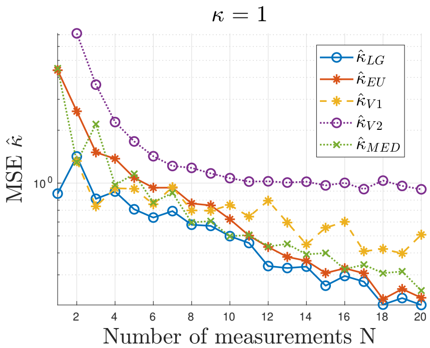

Fig. 5: Intrinsic MSE of each estimator for $\kappa = 1$

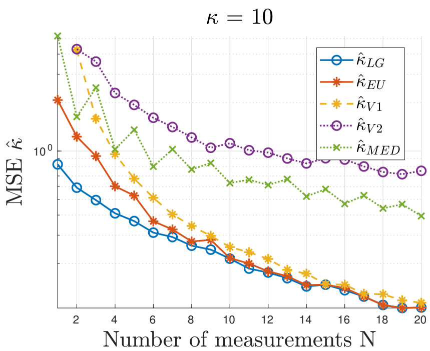

Fig. 6: Intrisic MSE of each estimator for $\kappa = 10$

V. Conclusions and Perspectives

In this communication, we introduce a novel approach for estimating the location parameter ϕ and the concentration parameter κ of the von Mises distribution. This method reformulates the estimation problem within a LG framework, providing a mathematically rigorous alternative to Euclidean techniques. The derived estimation algorithm on LGs has been validated through numerical simulations, comparing it with different methods from literature, and outperforms the Euclidean estimation algorithms for the estimation of the concentration parameter in harsh scenarios characterized by high variance (related with the C/N_0) and a small number of measurements. This last scenario description is desirable in the GNSS context since the receiver uses only one phase measurement at a time. With this approach, it could be possible to avoid time-correlation between two successive measurements created by phase lock loops. Then, it could improve tracking robustness within the GNSS receiver. On the other hand, such modeling could be applied in other fields, such as robotics or thermal noise characterization [24] [25].

Future work will focus on the analysis of dynamic cases where both the phase ϕ and the concentration parameter κ vary over time. This will involve the design of a Bayesian filter approach on Lie groups.

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