MISSPECIFIED TIME-DELAY AND DOPPLER ESTIMATION OVER NON GAUSSIAN SCENARIOS

Lorenzo Ortega [1,2], *Stefano Fortunati* [1,3],

[1] IPSA, [2] TéSA, [3] L2S

ABSTRACT

Time-delay and Doppler estimation is an operation performed in a plethora of engineering applications. A common hypothesis underlying most of the existing works is that the noise of the true and assumed signal model follows a centered complex normal distribution. However, everyday practice shows that the true signal model may differ from the nominal case and should be modeled by a non Gaussian distribution. In this paper, we analyse the asymptotic performance of the time-delay and Doppler estimation for the non-nominal scenario where the true noise model follows a centered complex elliptically symmetric (CES) distribution and the receiver assumed that the noise model follows a centered complex normal distribution. It turns out that performance bound under the misspecified model is equal to the one obtained for the well specified Gaussian scenario. In order to validate the theoretical outcomes, Monte Carlo simulations have been carried out.

***Index Terms*—** Complex elliptically symetric distribution, Misspecified Cramér-Rao bound, time-delay and Doppler estimation, band-limited signals.

1. INTRODUCTION

Time-delay and Doppler estimation is a central procedure in many engineering fields such as communications, radar or navigation [1–10], since it represents the first task of the receiver [5, 8, 9]. Due to its importance, it is of great practical interest to determine the ultimate achievable estimation performance in the mean squared error (MSE) sense. This information can be provided by the Cramér-Rao bound (CRB) [11,12], which is known to provide an accurate estimation of the MSE of the maximum likelihood estimator (MLE), in the asymptotic regime [13]. So, it is not surprising that several CRB expressions for the time-delay and Doppler estimation problem have been derived for the past decades, for finite narrow-band and wide-band signals [2,14–23]. In addition, several scenarios where the true and the assumed signal model at the receiver differ, have also been studied recently [24–27]. In those studies, expressions of the estimation limits provided by the Misspecified CRB (MCRB) [28,29] have been derived. Note that the MCRB provide us with the error covariance matrix of the MLE under model misspecfication, i.e. the Misspecified MLE (MMLE) [28, Theo. 2], [29, Sec. 4.4.3]. However, in all these previous works, a common hypothesis is that both the noise in the *true* signal model and the noise in the signal model *assumed* by the receiver follows a centered complex normal distribution. Nevertheless, to the best of the authors' knowledge, there is no reference in the state of the art which derives the ultimate achievable estimation performance of the time-delay and Doppler (in the MSE sense) for the case where the the true signal model is characterized by a non-Gaussian, complex elliptically symetric (CES)-distributed noise. The identification of these types of scenarios is of vital importance to characterized the loss of estimation performance of the time-delay and Doppler in case of non-nominal scenarios where unknown phenomena such as interference, multipath or cluttering could disrupt the reception of the emitted signal.

The contents of the study have been structured in five sections. Section 2 presents the true and the assumed signal models. Section 3 shows that the MMLE is an unbiased estimator by minimizing the Kullback-Leiber divergence between the true and assumed models. Section 4 derives the MCRB expressions and provides a closed-form MCRB expression of the parameters of interest considering a band-limited signal. The MCRB expresions are validated in Section 5 via Monte Carlo simulations for two representative CES distribution families. Finally, the key observations and contributions of the paper are collected in Section 6.

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2. TRUE AND MISSPECIFIED SIGNAL MODELS

2.1. True Signal Model

Let us consider that a band-limited signal $a(t)$, with bandwidth B, is transmitted over a carrier frequency f_c ($\lambda_c = c/f_c$, $\omega_c = 2\pi f_c$) from a transmitter T at position $\boldsymbol{P}_T(t)$ to a receiver R at position $\boldsymbol{P}_R(t)$. The distance travelled by the transmitted signal is $\boldsymbol{P}_{TR} = \|\boldsymbol{P}_T(t-\tau_0(t)) - \boldsymbol{P}_R(t)\| \approx (\boldsymbol{P}_T - \boldsymbol{P}_R) + vt$, that is, a first order approximation where $\bar{\tau} = \frac{(\boldsymbol{P}_T - \boldsymbol{P}_R)}{c}$ and $\bar{b} = \frac{v}{c}$ with v the relative velocity between the transmitter and the receiver. The received signal at the output of the Hilbert filter can be expressed as [14, 20, 30]

$$x\left(t;\bar{\boldsymbol{\eta}}\right) = \bar{\alpha} a\left(t-\bar{\tau}\right) e^{-j2\pi f_c\left(\bar{b}(t-\bar{\tau})\right)} + n\left(t\right), \tag{1}$$

where the narrowband assumption, i.e. the influence of the Doppler parameter on the baseband signal samples, is considered. Moreover, $n(t)$ is defined as a centered CES noise and $\bar{\alpha} = \bar{\rho} e^{j\bar{\Phi}}$ a complex gain. The discrete vector signal model is built from $N = N_1 - N_2 + 1$ samples at $T_s = 1/F_s = 1/B$,

$$\mathbf{x} = \bar{\alpha}\boldsymbol{\mu}(\bar{\boldsymbol{\eta}}) + \mathbf{n} = \bar{\rho} e^{j\bar{\Phi}}\boldsymbol{\mu}(\bar{\boldsymbol{\eta}}) + \mathbf{n}, \tag{2}$$

with $\mathbf{x} = \left(\ldots, x\left(kT_s\right), \ldots\right)^\top$, $N_1 \leq k \leq N_2$ signal samples,

$$\mathbf{n} = \left(\ldots, n\left(kT_s\right), \ldots\right)^\top, \tag{3}$$

with $n(kT_s) \sim CES(0, \bar{\sigma}_n^2, g)$, $\bar{\sigma}_n^2$ the variance of the noise, g the density generator [31]. We assume that the noise samples are i.i.d. Moreover, by posing $\bar{\boldsymbol{\eta}} = [\bar{\tau}, \bar{b}]^\top$, we have:

$$\boldsymbol{\mu}(\bar{\boldsymbol{\eta}}) = (\ldots, a(kT_s - \bar{\tau})e^{-j2\pi f_c(\bar{b}(kT_s-\bar{\tau})}, \ldots)^\top. \quad (4)$$

The *true* deterministic parameters can be gathered in vector $\bar{\boldsymbol{\epsilon}}^\top = (\bar{\sigma}_n^2, \bar{\rho}, \bar{\Phi}, \bar{\boldsymbol{\eta}}^\top) = (\bar{\sigma}_n^2, \bar{\boldsymbol{\theta}}^\top)$, with $\bar{\rho} \in \mathbb{R}^+, 0 \leq \bar{\Phi} \leq 2\pi$. The *true* and correctly specified signal model is represented by a probability densitiy function (pdf) denoted as $p_{\bar{\boldsymbol{\epsilon}}}(\mathbf{x}; \bar{\boldsymbol{\epsilon}}) = \Pi_{k=N_1}^{N_2} p_{\bar{\boldsymbol{\epsilon}}}(x_k, \bar{\boldsymbol{\epsilon}})$, with $p_{\bar{\boldsymbol{\epsilon}}}(x_k, \bar{\boldsymbol{\epsilon}}) = CES(\bar{\alpha}\boldsymbol{\mu}_k(\bar{\boldsymbol{\eta}}), \bar{\sigma}_n^2, g)$.

2.2. Misspecified Signal Model

In standard receivers, the noise is usually considered to be centered complex normal and the MLE associated with this signal model is usually implemented. This nominal case leads to the definition of the *misspecified* parameter vector $\boldsymbol{\eta} = [\tau, b]^\top$, and the complete set of unknown *misspecifed* parameters $\boldsymbol{\epsilon}^\top = [\sigma_n^2, \rho, \Phi, \boldsymbol{\eta}^\top] = [\sigma_n^2, \boldsymbol{\theta}^\top]$, yielding the following misspecified signal model at the output of the Hilbert filter,

$$x(t; \boldsymbol{\eta}) = \alpha a(t - \tau)e^{-j2\pi f_c b(t-\tau)} + n(t), \quad (5)$$

with $n(t)$ is assumed to be a complex white Gaussian noise with unknown variance σ_n^2 and $\alpha = \rho e^{j\Phi}$. Again, we can build the discrete vector signal model from N i.i.d. samples at T_s as:

$$\mathbf{x} = \alpha\boldsymbol{\mu}(\boldsymbol{\eta}) + \mathbf{n}, \quad (6)$$

$$\boldsymbol{\mu}(\boldsymbol{\eta}) = (\ldots, a(kT_s - \tau)e^{-j2\pi f_c b(kT_s-\tau)}, \ldots)^\top. \quad (7)$$

The misspecified signal model is represented by a pdf denoted as $f_{\boldsymbol{\epsilon}}(\mathbf{x}; \boldsymbol{\epsilon})$ which follows a complex circular Gaussian distribution $\mathbf{x} \sim \mathcal{CN}(\alpha\boldsymbol{\mu}(\boldsymbol{\eta}), \sigma_n^2 \boldsymbol{I}_N)$. From now on, we denote the MLE of an estimator under a misspecified signal model as MMLE. This mismatched estimator is generally biased as it has been verified under certain conditions. Please refers to [24–26]. Moreover, these biased estimated parameters are commonly referred to as pseudotrue parameters, $\boldsymbol{\epsilon}_{pt}^\top = [\sigma_{pt}^2, \rho_{pt}, \Phi_{pt}, \tau_{pt}, b_{pt}]$.

3. PSEUDOTRUE PARAMETERS COMPUTATION VIA KULLBACK-LEIBLER DIVERGENCE

The pseudotrue parameters are simply those that give the minimum Kullback-Leibler Divergence (KLD) [28, 29] $D(p_{\bar{\boldsymbol{\epsilon}}}||f_{\boldsymbol{\epsilon}}) = E_{p_{\bar{\boldsymbol{\epsilon}}}}\left[\ln p_{\bar{\boldsymbol{\epsilon}}}(\mathbf{x}; \bar{\boldsymbol{\epsilon}}) - \ln f_{\boldsymbol{\epsilon}}(\mathbf{x}; \boldsymbol{\epsilon})\right]$, between the true and assumed models, where $E_{p_{\bar{\boldsymbol{\epsilon}}}}[\cdot]$ is the expectation with respect to (w.r.t.) the true model's pdf,

$$\boldsymbol{\epsilon}_0 = \arg\min_{\boldsymbol{\epsilon}} \{D(p_{\bar{\boldsymbol{\epsilon}}}||f_{\boldsymbol{\epsilon}})\} = \arg\min_{\boldsymbol{\epsilon}} \left\{E_{p_{\bar{\boldsymbol{\epsilon}}}}\left[-\ln f_{\boldsymbol{\epsilon}}(\mathbf{x}; \boldsymbol{\epsilon})\right]\right\}, \quad (8)$$

$$E_{p_{\bar{\boldsymbol{\epsilon}}}}\left[-\ln f_{\boldsymbol{\epsilon}}\right] = -N\ln(\pi) - E_{p_{\bar{\boldsymbol{\epsilon}}}}\left[N\ln(\sigma_n^2)\right] + E_{p_{\bar{\boldsymbol{\epsilon}}}}\left[\frac{(\mathbf{x}-\alpha\boldsymbol{\mu}(\boldsymbol{\eta}))^H(\mathbf{x}-\alpha\boldsymbol{\mu}(\boldsymbol{\eta}))}{\sigma_n^2}\right]. \quad (9)$$

To compute $\boldsymbol{\epsilon}_0^\top = [\sigma_0^2, \rho_0, \Phi_0, \tau_0, b_0]$, we have to minimize (8) w.r.t. the argument $\boldsymbol{\epsilon}$:

$$\arg\min_{\boldsymbol{\epsilon}} \left\{E_{p_{\bar{\boldsymbol{\epsilon}}}}\left[-\ln f_{\boldsymbol{\epsilon}}(\mathbf{x}; \boldsymbol{\epsilon})\right]\right\} \quad (10)$$
$$= \arg\min_{\boldsymbol{\epsilon}} \left\{E_{p_{\bar{\boldsymbol{\epsilon}}}}\left[\frac{1}{\sigma_n^2}\left[\|\mathbf{x} - \alpha\boldsymbol{\mu}(\boldsymbol{\eta})\|^2\right]\right] - E_{p_{\bar{\boldsymbol{\epsilon}}}}\left[N\ln(\sigma_n^2)\right]\right\}$$

Let us start by minimizing w.r.t. to the variance σ_n^2:

$$E_{p_{\bar{\boldsymbol{\epsilon}}}}\left[\frac{\partial}{\partial\sigma_n^2}\ln f_{\boldsymbol{\epsilon}}(\mathbf{x}; \boldsymbol{\epsilon})\right] = E_{p_{\bar{\boldsymbol{\epsilon}}}}\left[\frac{N}{\sigma_n^2} - \frac{1}{\sigma_n^4}\|\mathbf{x} - \alpha\boldsymbol{\mu}(\boldsymbol{\eta})\|^2\right]$$
$$= \frac{N}{\sigma_n^2} - \frac{N}{\sigma_n^4}\bar{\sigma}_n^2 \quad (11)$$

and $\sigma_0^2 = \bar{\sigma}_n^2$. On the other hand, the minimization with respect to the parameters is simplified to

$$\arg\min_{\boldsymbol{\theta}} \left\{E_{p_{\bar{\epsilon}}}\left[-\ln f_{\boldsymbol{\epsilon}}(\mathbf{x}; \boldsymbol{\epsilon})\right]\right\} \quad (12)$$
$$= \arg\min_{\boldsymbol{\theta}} \left\{E_{p_{\bar{\epsilon}}}\left[\|\mathbf{x} - \alpha\boldsymbol{\mu}(\boldsymbol{\eta})\|^2\right]\right\}$$
$$= \arg\min_{\boldsymbol{\theta}} \left\{\left[\|\bar{\alpha}\boldsymbol{\mu}(\bar{\boldsymbol{\eta}}) - \alpha\boldsymbol{\mu}(\boldsymbol{\eta})\|^2\right]\right\}$$

and $\boldsymbol{\theta}_0^\top = [\rho_0, \Phi_0, \tau_0, b_0] = \bar{\boldsymbol{\theta}}^\top = [\bar{\rho}, \bar{\Phi}, \bar{\tau}, \bar{b}]$. Remarkably, this shows that the pseudo-true parameter is equal to the true one and consequently, according to [28, Theo. 2], [29, Sec. 4.4.3], the MMLE result to be consistent and asymptotically unbiased.

4. CLOSED FORM EXPRESSION FOR THE MCRB

In this section, we derive the MCRB from the parameters of interest $\boldsymbol{\epsilon}_0^\top = \bar{\boldsymbol{\epsilon}}^\top$. This result can be derived from the general formula introduced in [32], where it is shown that the MCRB can be calculated from the product of two matrices $\mathbf{A}$ and $\mathbf{B}$ as follows:

$$\mathbf{MCRB}(\boldsymbol{\epsilon}_0) = \mathbf{MCRB}(\bar{\boldsymbol{\epsilon}}) = \mathbf{A}(\bar{\boldsymbol{\epsilon}})^{-1}\mathbf{B}(\bar{\boldsymbol{\epsilon}})\mathbf{A}(\bar{\boldsymbol{\epsilon}})^{-1}, \quad (13)$$

Proof: Since, as shoved in Sec. 3, the pseudo-true parameter vector is equal to the true parameter vector, i.e. $\boldsymbol{\epsilon}_0 = \bar{\boldsymbol{\epsilon}}$, in the following derivation we will only use the true parameter vector $\boldsymbol{\epsilon}$. The proof of (13) can be obtained by specializing the results presented in [32, Sec. 3.2]. Specifically, we have a set of N uni-variate i.i.d observations $\{x_k\}_{k=N_1}^{N_2}$, such that $x_k \sim p_{\bar{\boldsymbol{\epsilon}}}(x_k, \bar{\boldsymbol{\epsilon}}) = CES(\bar{\alpha}\boldsymbol{\mu}_k(\bar{\boldsymbol{\eta}}), \bar{\sigma}_n^2, g)$. Consequently, from the Stochastic Representation Theorem [31, Theo. 3]:

$$x_k =_d \bar{\alpha}\boldsymbol{\mu}_k(\bar{\boldsymbol{\eta}}) + \sqrt{\mathcal{Q}}\bar{\sigma}_n u_k, \quad (14)$$

where u_k is a complex uni-variate random variable uniformly distributed on $\mathbb{C}S \triangleq \{u \in \mathbb{C}||u| = 1\}$, i.e. $u_k \sim U(\mathbb{C}S)$. The *second order modular variate* $\mathcal{Q}$ is a positive random variable, independent from u_k with pdf $p_{\mathcal{Q}}(q) = \delta_g^{-1}g(q)$, where $\delta_g \triangleq \int_0^\infty g(q)dq$ is a normalizing constant (see [31, Eq. (19)]). To avoid the well-known scale ambiguity between $\bar{\sigma}_n$ and g, we impose that $E\{\mathcal{Q}\} = 1$. Note that, this constraint allows us to consider $\bar{\sigma}_n$ as the *statistical power* P of the data x_k, (see the discussion in [31, Sec. III.C]), since from (14), we have that:

$$P \triangleq E\{|x_k - \bar{\alpha}\boldsymbol{\mu}_k(\bar{\boldsymbol{\eta}})|^2\} = E\{\mathcal{Q}\}E\{|u_k|^2\}\bar{\sigma}_n^2 = \bar{\sigma}_n^2, \quad (15)$$

since $E\{|u_k|^2\} = 1$ [31, Lemma 1].

According to the misspecified signal model introduced in Sec. 2.2, the assumed pdf is given by $f_{\boldsymbol{\epsilon}}(\mathbf{x}; \boldsymbol{\epsilon}) = \mathcal{CN}(\alpha\boldsymbol{\mu}(\boldsymbol{\eta}), \sigma_n^2\mathbf{I}_N)$. This is a particular case of the Scenario 1 in [32, Sec. 3.2] and consequently the matrices $\mathbf{A}$ and $\mathbf{B}$, needed to evaluate the MCRB, can be obtained from eq. [32, Eq. (34)] and [32, Eq. (30)], respectively. To this end, we have the following simplifications:

- S1 The bias vector $\mathbf{r}_k^0$ in [32, Eq. (32)] is nil, since $\boldsymbol{\epsilon}_0 = \bar{\boldsymbol{\epsilon}}$.
- S2 The matrix $\mathbf{P}_i^0$ in [32, Eq. (19)] is nil for $i \in \{2, 3, 4, 5\}$ while $\mathbf{P}_1^0 \equiv P_1^0 = \sigma_n^{-2}$.
- S3 The matrix $\mathbf{P}_{ij}^0$ in [32, Eq. (25)] is nil $\forall i, j \in \{1, 2, 3, 4, 5\}$.

S4 The matrix $\mathbf{S}_i^0$ in [32, Eq. (23)] is nil for $i \in \{2,3,4,5\}$ while $\mathbf{S}_1^0 \equiv S_1^0 = \bar{\sigma}_n^{-4}$.

S5 The term $\mathrm{tr}(\mathbf{S}_i^0\mathbf{\Sigma})$ is nil for $i \in \{2,3,4,5\}$ while $\mathrm{tr}(\mathbf{S}_1^0\mathbf{\Sigma}) = \mathrm{tr}(S_1^0\sigma_n^2) = \bar{\sigma}_n^{-2}$.

S6 By indicating as $\delta(\cdot)$ the Kronecker delta function, we have that $\mathrm{tr}(\mathbf{S}_i^0\mathbf{\Sigma}\mathbf{S}_j^0\mathbf{\Sigma}) = \bar{\sigma}_n^{-4}\delta(i-1)\delta(j-1)$, $\forall i,j \in \{1,2,3,4,5\}$.

By exploiting S1 - S6, it is immediate to verify that the term $\alpha_k^{ij}(\bar{\boldsymbol{\epsilon}})$ in [32, Eq. (33)] is given by:

$$\alpha_k^{ij}(\bar{\boldsymbol{\epsilon}}) = \frac{2}{\bar{\sigma}_n^2}\Re\left\{\left[\frac{\partial(\bar{\alpha}\boldsymbol{\mu}_k(\bar{\boldsymbol{\eta}}))}{\partial\epsilon_i}\right]^H \frac{\partial(\bar{\alpha}\boldsymbol{\mu}_k(\bar{\boldsymbol{\eta}}))}{\partial\epsilon_j}\right\} + \\ + 2\bar{\sigma}_n^{-4}\delta(i-1)\delta(j-1), \tag{16}$$

By substituting $\alpha_k^{ij}(\bar{\boldsymbol{\epsilon}})$ in the general expression of the matrix $\mathbf{A}(\bar{\boldsymbol{\epsilon}})$ given in [32, Eq. (34)], we get:

$$[\mathbf{A}(\bar{\boldsymbol{\epsilon}})]_{i,j} = -\frac{2}{\bar{\sigma}_n^2}\sum_{k=N_1}^{N_2}\Re\left\{\left[\frac{\partial(\bar{\alpha}\boldsymbol{\mu}_k(\bar{\boldsymbol{\eta}}))}{\partial\epsilon_i}\right]^H \frac{\partial(\bar{\alpha}\boldsymbol{\mu}_k(\bar{\boldsymbol{\eta}}))}{\partial\epsilon_j}\right\} + \\ - N\bar{\sigma}_n^{-4}\delta(i-1)\delta(j-1), \tag{17}$$

or, in matrix form:

$$\mathbf{A}(\bar{\boldsymbol{\epsilon}}) = \begin{pmatrix} -N\bar{\sigma}_n^{-4} & \mathbf{0}_{1\times4} \\ \mathbf{0}_{4\times1} & -\mathbf{C}(\bar{\boldsymbol{\theta}}) \end{pmatrix} \tag{18}$$

where the matrix $\mathbf{C}(\bar{\boldsymbol{\theta}})$ is given by:

$$\mathbf{C}(\bar{\boldsymbol{\theta}}) = \frac{2}{\bar{\sigma}_n^2}\sum_{k=N_1}^{N_2}\Re\left\{\left[\frac{\partial(\bar{\alpha}\boldsymbol{\mu}_k(\bar{\boldsymbol{\eta}}))}{\partial\boldsymbol{\theta}}\right]^H \frac{\partial(\bar{\alpha}\boldsymbol{\mu}_k(\bar{\boldsymbol{\eta}}))}{\partial\boldsymbol{\theta}}\right\} \tag{19}$$

By making use of S1 - S6, the general expression of the matrix $\mathbf{B}(\bar{\boldsymbol{\epsilon}})$ given in [32, Eq. (31)] can be simplified as:

$$[\mathbf{B}(\bar{\boldsymbol{\epsilon}})]_{i,j} = \frac{2}{\bar{\sigma}_n^2}\Re\left\{\left[\frac{\partial(\bar{\alpha}\boldsymbol{\mu}_k(\bar{\boldsymbol{\eta}}))}{\partial\epsilon_i}\right]^H \frac{\partial(\bar{\alpha}\boldsymbol{\mu}_k(\bar{\boldsymbol{\eta}}))}{\partial\epsilon_j}\right\} + \\ + N\bar{\sigma}_n^{-4}(E\{\mathcal{Q}^2\}-1)\delta(i-1)\delta(j-1). \tag{20}$$

or, in matrix form:

$$\mathbf{B}(\bar{\boldsymbol{\epsilon}}) = \begin{pmatrix} N\bar{\sigma}_n^{-4}(E\{\mathcal{Q}^2\}-1) & \mathbf{0}_{1\times4} \\ \mathbf{0}_{4\times1} & \mathbf{C}(\bar{\boldsymbol{\theta}}) \end{pmatrix}. \tag{21}$$

Finally, the MCRB can be expressed as:

$$\mathbf{MCRB}(\bar{\boldsymbol{\epsilon}}) = \mathbf{A}(\bar{\boldsymbol{\epsilon}})^{-1}\mathbf{B}(\bar{\boldsymbol{\epsilon}})\mathbf{A}(\bar{\boldsymbol{\epsilon}})^{-1} \\ = \begin{pmatrix} \frac{\bar{\sigma}_n^4}{N}(E\{\mathcal{Q}^2\}-1) & \mathbf{0}_{1\times4} \\ \mathbf{0}_{4\times1} & \left[\mathbf{C}(\bar{\boldsymbol{\theta}})\right]^{-1} \end{pmatrix}. \tag{22}$$

We note, in passing that if the true distribution is a Gaussian one, i.e. $x_k \sim p_{\bar{\boldsymbol{\epsilon}}}(x_k, \bar{\boldsymbol{\epsilon}}) = \mathcal{CN}(\bar{\alpha}\boldsymbol{\mu}_k(\bar{\boldsymbol{\eta}}), \bar{\sigma}_n^2)$, the term $E\{\mathcal{Q}^2\}$ is equal to 2 as proved in [32, Eq. (41)] and this lead us to the classical result about the CRB on the estimation of the variance in complex Gaussian data. In any of the cases, the matrix linked to the parameters of interest $\boldsymbol{\theta}$ yields to the classical Fisher Information Matrix (FIM) of the Gaussian scenario, proving that asymptotic estimation performance of $\boldsymbol{\theta}$ are similar to the well specified Gaussian model. This surprinsing result can be explained using the semiparametric theory (see [33, Sec. IV.B] and [34, Sec. III.B]). Due to the space limitation, in deep discussons on this point are left to future works.

4.1. Closed-Form MCRB Expression for a Band-Limited Signal

It is interesting to note that the matrix $\boldsymbol{C}(\bar{\boldsymbol{\theta}})$ in (19) represents the FIM of a single source conditional signal model (CSM) [13]. A compact expression of this FIM, that depends only on the baseband signal samples, was recently derived in [20] as:

$$\mathbf{C}(\bar{\boldsymbol{\theta}}) = \frac{2F_s}{\sigma_n^2}\Re\left\{\mathbf{Q}\mathbf{W}\mathbf{Q}^H\right\} = \mathbf{MCRB}^{-1}(\boldsymbol{\theta}_0) \tag{23}$$

with

$$\mathbf{W} = \begin{bmatrix} w_1 & w_2^* & w_3^* \\ w_2 & W_{2,2} & w_4^* \\ w_3 & w_4 & W_{3,3} \end{bmatrix}, \tag{24a}$$

$$\mathbf{Q} = \begin{bmatrix} e^{j\bar{\Phi}} & 0 & 0 \\ j\bar{\alpha} & 0 & 0 \\ j\bar{\alpha}\omega_c\bar{b} & 0 & -\bar{\alpha} \\ 0 & -j\bar{\alpha}\omega_c & 0 \end{bmatrix}, \tag{24b}$$

where the elements of $\mathbf{W}$ can be expressed w.r.t. the baseband signal samples as,

$$w_1 = \frac{1}{F_s}\mathbf{a}^H\mathbf{a}, \quad w_2 = \frac{1}{F_s^2}\mathbf{a}^H\mathbf{D}\mathbf{a}, \quad w_3 = \mathbf{a}^H\mathbf{\Lambda}\mathbf{a}, \tag{25}$$
$$w_4 = \frac{1}{F_s}\mathbf{a}^H\mathbf{D}\mathbf{\Lambda}\mathbf{a}, \quad W_{2,2} = \frac{1}{F_s^3}\mathbf{a}^H\mathbf{D}^2\mathbf{a}, \quad W_{3,3} = F_s\mathbf{a}^H\mathbf{V}\mathbf{a}.$$

with $\mathbf{a}$, the baseband samples vector, $\mathbf{D}$, $\mathbf{\Lambda}$ and $\mathbf{V}$ defined as,

$$\mathbf{a} = (\ldots, a(nT_s), \ldots)^\top_{N_1 \le n \le N_2}, \tag{26a}$$
$$\mathbf{D} = \mathrm{diag}(\ldots, n, \ldots)_{N_1 \le n \le N_2}, \tag{26b}$$
$$(\mathbf{\Lambda})_{n,n'} = \left| \begin{array}{l} n' \neq n : \frac{(-1)^{|n-n'|}}{n-n'} \\ n' = n : 0 \end{array} \right. \tag{26c}$$
$$(\mathbf{V})_{n,n'} = \left| \begin{array}{l} n' \neq n : (-1)^{|n-n'|}\frac{2}{(n-n')^2} \\ n' = n : \frac{\pi^2}{3} \end{array} \right. \tag{26d}$$

5. SIMULATIONS

We consider two possible scenarios where a GPS L1 C/A signal [10] is received by a GNSS receiver which assumes that the noise follows a standard centered normal distribution. **Scenario 1** In the first scenario, we set a true signal model where the noise is distributed according to a complex centered Generalized Gaussian (GG) distribution , [29, Sec. 4.6.1.2] with exponent $s > 0$ and scale $b > 0$, where s is a parameter controlling the level of non-Gaussianity. The second-order modular variate $\mathcal{Q}$ of a GG distribution is given by $\mathcal{Q} =_d G^{1/s}$ where G is a Gamma distributed random variable with parameter $1/s$ and b, i.e. $G \sim \mathrm{Gam}(1/s, b)$ [31, Sec. IV.B]. In order to satisfy the constraint $E\{\mathcal{Q}\} = 1$ (see section 4), we set $b = \left(\frac{\bar{\sigma}_n^2\Gamma(1/s)}{\Gamma(2/s)}\right)^s$ where $\bar{\sigma}_n^2$ depends on the signal to noise ratio at the output of the match filter SNR_{out}. The SNR_{out} is defined as:

$$SNR_{out} = \frac{|\bar{\alpha}|^2\mathbf{a}^H\mathbf{a}}{\bar{\sigma}_n^2}. \tag{27}$$

Scenario 2 In a second scenario, we set a true signal model where the noise is sampled from a complex centered t-distribution [29,

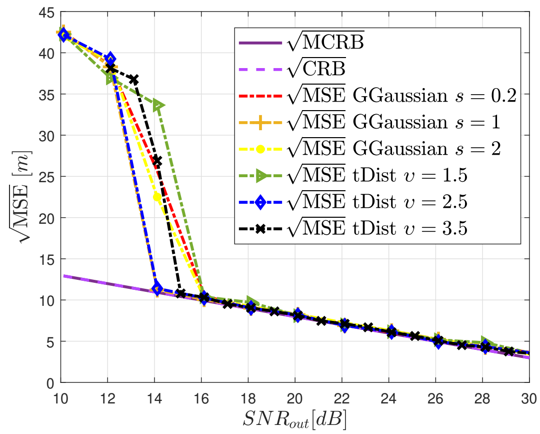

Fig. 1: RMSE of the MMLE of the time-delay considering complex centered GG dist. with $s = \{0.2, 1, 2\}$ and complex centered t-dist. with $\upsilon = \{1.5, 2.5, 3.5\}$. The sampling frequency is set to $F_s = 4$ MHz and the integration time is set to $T = 1$ms.

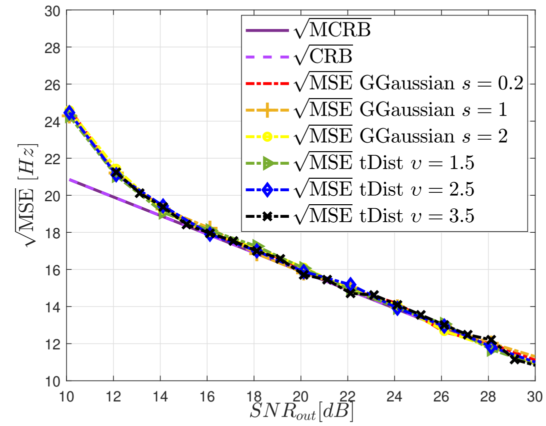

Fig. 2: RMSE of the MMLE of the Doppler considering complex centered GG dist. with $s = \{0.2, 1, 2\}$ and complex centered t-dist. with $\upsilon = \{1.5, 2.5, 3.5\}$. The sampling frequency is set to $F_s = 4$ MHz and the integration time is set to $T = 1$ms.

Sec. 4.6.1.1] with $\upsilon > 1$ degrees of freedom (or *shape parameter*) that control the level of non-Gaussianity and a scale parameter μ. The second-order modular variate $\mathcal{Q}$ of a t-distribution is an F-distributed random variable with parameter 2 and υ i.e. $\mathcal{Q} \sim F(2, \upsilon)$ [31, Sec. IV.A]. Then, in order to meet the constraint $E\{\mathcal{Q}\} = 1$, the scale as to be set as $\mu = \frac{\upsilon}{\bar{\sigma}_n^2(\upsilon-1)}$. Again, $\bar{\sigma}_n^2$ set the SNR_{out}. As discussed in [27], the MMLE for the joint estimation of the time-delay and Doppler can be expressed as: [1]

$$\hat{\boldsymbol{\eta}} = \arg\max_{\boldsymbol{\eta}} \left\| \boldsymbol{\Pi}_{\boldsymbol{\mu}(\boldsymbol{\eta})} \boldsymbol{x} \right\|^2 \tag{28}$$

The root mean square error (RMSE) results of the MMLE for the parameters of interest $\boldsymbol{\eta}^T = [\tau, b]$ are shown in Figs. 1 and 2 w.r.t. the SNR_{out} and considering the following setup: a GNSS receiver with sampling frequency $F_s = 4$ MHz and an integration time of 1 ms. The number of Monte Carlo is set to 1000 iterations. In the simulation, complex centered Generalized Gaussian distributions with $s = \{0.2, 1, 2\}$ and complex centered t-distributions with $\upsilon = \{1.5, 2.5, 3.5\}$ have been used as a true model. In the results one can observe that the RMSE ($\sqrt{MSE}$) of the pseudotrue parameter converges to the asymptotic estimation performance derived in Section 4. These results confirm the theoretical derivation. Note also that the $\sqrt{MCRB}$ is equal to the $\sqrt{CRB}$. It is worth to underline that the previous theoretical results are valid for all the CES-distributed true noise model and not only of the GG and the t-distribution. As said before, a formal explanation of this fact relies on the semiparametric theory (see [33, Sec. IV;B] and [34, Sec. III.B]) and an comprehensive explanation will be provided in future works. Here, due to the space limitation, we limit ourselves to this observation: the equality between the MCRB and the CRB is verified only when the parameters of interest parameterize the mean of the observation vectors. On the other hand, if the parameters of interest parameterise the covariance matrix of the observations, the equality may be no longer valid and the MMLE may fail to be a consistent estimator of the true parameters.

[1] Let $S = span(\mathbf{A})$, with $\mathbf{A}$ a matrix, be the linear span of the set of its column vectors. The orthogonal projector over S is $\boldsymbol{\Pi}_{\mathbf{A}} = \mathbf{A}\left(\mathbf{A}^H\mathbf{A}\right)\mathbf{A}^H$.

6. CONCLUSION

The purpose of this paper was to introduce new inputs on the time-delay and Doppler estimation theory. In particular, we have derived the asymptotic performance expressions (MCRB) for the non-nominal scenario where the true noise model follows a centered CES distribution and the receiver assumed that the noise model follows a centered complex normal distribution. The theoretical results have been validated by means of Monte Carlo simulations. Remarkably, two counter intuitive results have been proved: $i)$ the MMLE for the estimation of time-delay and Doppler is consistent w.r.t. the true parameters even in the presence of a wrong Gaussian assumption and $ii)$ the MCRB is equal to the CRB derived under a nominal Gaussian scenario. This is a fundamental result since it tells us that we can continue to use the MLE in (28), derived under a Gaussian data assumption, even in the presence of a non-Gaussian, CES-distributed noise *without any loss in the MSE sense*. The foundation of this important result is to be sought in semiparametric theory and will be the subject of future work.

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