SELECTIVE ANALYTIC SIGNAL CONSTRUCTION FROM A NON-UNIFORMLY SAMPLED BANDPASS SIGNAL

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ABSTRACT

This paper proposes a method that simultaneously builds the analytic signal from non-uniform samples of a bandpass signal and rejects interferences. The analytic signal is required for many onboard operations in communication satellites. This method operates in the time domain and without preliminary demodulation, using Periodic Non-uniform Sampling of order 2 (PNS2). This non-uniform sampling scheme can be easily implemented with available devices. Exact formulas for the analytic signal construction are derived for an infinite observation window (an infinite number of samples). For practical applications, the formulas should also demonstrate a high convergence rate due to the finite observation window. Formulas with increasing convergence rates are thus derived. The proposed method has been tested through simulations according to the number of available samples, the interference parameters and the filter transfer function regularity.

***Index Terms*—** Signal processing, Non-uniform signal sampling, Signal reconstruction, Analog-digital conversion

1. INTRODUCTION AND PROBLEM FORMULATION

In many applications, the signal of interest can be modeled as a bandpass signal corrupted by bandpass interferences. This is the case in multi-user communications using Frequency Division Multiple Access (FDMA). For this application, the reduction of adjacent channel interference is a key problem [1]. Moreover, the in phase and quadrature components and thus the analytic signal are required for many onboard operations in communication satellites such as filtering, multiplexing, demultiplexing or beamforming [2]. This paper focuses on the construction of the analytic signal from non-uniform samples of the original bandpass signal with simultaneous interference cancellation. The method uses a particular sampling scheme called Periodic Non-uniform Sampling of order 2 (PNS2), a particular case of PNSL ([3], [4], [5], [6], [7]) that uses L interleaved uniform sample sequences expressed as $\{nT_S + \tau_i,\ n \in \mathbb{Z}\}$ with $i \in \{1, ..., L\}$ and temporal delay $\tau_i \in \mathbb{R}$. Each sequence has the same rate $f_S = \frac{1}{T_S}$ chosen according to the Landau criterion [8]. This scheme allows to reduce sampling frequency ([9], [10], [11]) but also aliasing ([3], [7]). An exact reconstruction is possible from an infinite number of samples assuming the *a priori* knowledge of the signal spectral band [12]. However, in practical applications, the signal reconstruction is generally performed in real time using a finite sliding observation window or equivalently a finite number of samples. Thus the reconstruction formulas should demonstrate a high convergence rate towards the exact reconstruction [11], [13], [14]. Formulas with increasing convergence rates are obtained by introducing filters with increasingly regular transfer functions in [15]. In the present paper, new constraints are imposed to these filters in order to remove the interference and to build the analytic signal at the same time. The proposed method does not require any preliminary demodulation since it operates on the bandpass signal with a sub-Nyquist rate chosen according to the Landau criterion. Moreover, the simultaneous filtering and analytic signal construction are performed in the time domain using explicit expressions of the interpolation functions. This paper is organized as follows. Sections 1.1 and 1.2 present the signal model, the sampling scheme and classical reconstruction formulas. Relation to prior work is discussed in Section 2. Section 3 details the selective analytic signal construction formulas and their convergence properties. Section 4 studies the performance of the proposed method, tested through simulations according to the number of available samples, the filter transfer function regularity and the interference parameters. Conclusion and future works are reported in Section 5.

1.1. Signal model

For generality and further application to telecommunications, the signal of interest is modeled as a stationary random process $\mathbf{X} = \{X(t),\ t \in \mathbb{R}\}$ with zero mean, finite variance and power spectral density $s_X(f)$ defined by:

$$\mathrm{E}[X(t)X^*(t-\tau)] = \int_{\Delta} e^{2i\pi f\tau} s_X(f)\, df \qquad (1)$$

This work has been carried out under the financial support of the CNES (French government spatial agency) and Thales Alenia Space.

where E[·] stands for the mathematical expectation and the superscript $*$ for the complex conjugate. Moreover, the signal is bandpass with spectral band $\mathcal{B}_X = \mathcal{B}_X^- \cup \mathcal{B}_X^+ = (-F_2, -F_1) \cup (F_1, F_2)$. Some additional notations must be defined for further formula simplifications: $F_m = \frac{F_1+F_2}{2}$ and $\Delta = \frac{F_2-F_1}{2} \geq 0$. Now, let assume that $\frac{2k-1}{2} < F_1 \leq F_2 < \frac{2k+1}{2}$ for a given $k \in \mathbb{N}$. In other words, the signal spectral band is included in the so-called k^{th} Nyquist band defined in terms of normalized frequency by:

$$\mathcal{B}_{\mathcal{N}}(k) = \left(-k - \frac{1}{2}, -k + \frac{1}{2}\right) \cup \left(k - \frac{1}{2}, k + \frac{1}{2}\right). \quad (2)$$

This hypothesis is required for exact reconstruction from PNS2 sampling. The method proposed in this paper applies to both lowpass ($k = 0$) and bandpass signals ($k \neq 0$). However, this paper more specifically focuses on bandpass signals for potential satellite communication applications. The onboard received signal is the sum of the signal of interest and of an interference signal with spectral band $\mathcal{B}_{\text{int}}$ included in $\mathcal{B}_{\mathcal{N}}(k) \setminus \mathcal{B}_X$. In other words, the signal of interest and the interference are supposed to have non-overlapping supports included in the same Nyquist band as schemed in Figure 1. Under this necessary condition and in the case of an infinite observation window, this paper proposes errorless formulas for the construction of the analytic signal with total cancellation of the interference.

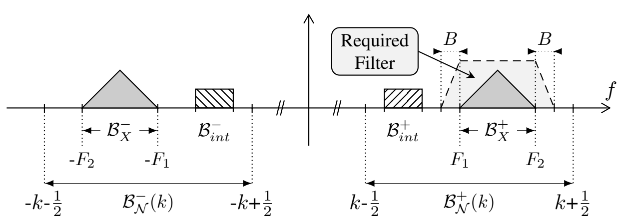

Fig. 1. Band schemes and required filter

1.2. PNS2 sampling scheme and reconstruction formulas

In the case of the PNS2 sampling scheme, the sample times are distributed according to two time interleaved periodic sequences $\{n+a, n \in \mathbb{Z}\}$ and $\{n+b, n \in \mathbb{Z}\}$ with $a, b \in [0, 1[$. The associated mean sampling rate equals 2 and thus fits the signal bandwidth (according to the Landau criterion). This allows to overcome the Nyquist criterion related to the maximum frequency. Signal samples are then composed of the sequences $\mathbf{X}_\lambda = \{X(n+\lambda), n \in \mathbb{Z}\}$ with $\lambda = \{a, b\}$. For bandpass signals (in the k^{th} Nyquist band with $k > 0$), under the condition $2k(b-a) \notin \mathbb{Z}$, exact signal reconstruction in mean-squared error sense and from an infinite number of samples is possible using the following formula [12], [15]:

$$X(t) = \frac{A_a(t)\sin[2\pi k(t-b)] + A_b(t)\sin[2\pi k(a-t)]}{\sin[2\pi k(a-b)]},$$
$$\text{with:} \quad A_\lambda(t) = \sum_{n\in\mathbb{Z}} \frac{\sin[\pi(t-n-\lambda)]}{\pi(t-n-\lambda)} X(n+\lambda). \quad (3)$$

Fast convergence reconstruction formulas have been studied in [15] for practical applications where only a finite number of samples is available. However these formulas lead to the reconstruction of the signal (and not of the analytic signal) on the entire band $\mathcal{B}_{\mathcal{N}}(k)$ that means in our case to the reconstruction of the interference also. The method proposed in this paper overcomes this problem and has a double purpose. First it constructs the analytic signal related to the signal of interest, allowing possible onboard processing such as complex filtering or beamforming. Second this construction is performed only on a specific sub-band of the considered Nyquist band $\mathcal{B}_{\mathcal{N}}(k)$, i.e. in our case on $\mathcal{B}_X$ which allows the interference cancellation. The construction of the analytic signal and the interference rejection are performed in a single step with fast convergence formulas adapted to practical applications.

2. RELATION TO PRIOR WORK

Use of PNS2 sampling to process bandpass signal reconstruction has been widely studied since Kohlenbergs formulation in [3]. This particular sampling scheme allows to reduce the effective sampling frequency [9], [10], [11] but also aliasing [3], [7]. This scheme has thus the advantage to be easily implementable with available devices as it requires two interleaved uniform sequences. In some cases it is used to model Analog-to-Digital Converters synchronization errors classified in the mismatch errors [16]. This paper proposes modified versions of errorless reconstruction of a bandpass signal from PNS2 for practical applications in telecommunication satellites. Errorless reconstruction from PNS2 has been previously studied in [12], [17]. In the context of telecommunications systems reconstruction formulas must also demonstrate high convergence rate due to the limited number of available samples. Convergence rate improvement has been studied in the literature [11], [13], [14]. However, the method proposed in [18] is original since it is based on a linear filtering operating in the time domain. This preliminary work leads to the main idea developed here: performing the reconstruction of the filtered signal of interest. From that idea, the filtering operation leads us to perform in a single step the convergence rate improvement [15], interference rejection interesting in FDMA applications [1] and also analytic signal construction required for onboard operations in communication satellites [2]. As far as we know such a method has never been studied, particularly in the case of bandpass signals.

3. HIGH CONVERGENCE RATE SELECTIVE ANALYTIC SIGNAL CONSTRUCTION

3.1. Principle

Selective construction of the analytic signal located in a given spectral band is performed using the PNS2 sampling scheme and modified reconstruction formulas. In this paper, such formulas are derived in order to integrate an appropriate filtering operation. The idea is to reconstruct the signal of interest filtered in an appropriate way. In this purpose, the reconstruction formula is modified to include a linear time-invariant filter with appropriate regularity properties (for convergence rate improvement [15]), with a bandwidth adapted to the signal bandwidth (for selective reconstruction i.e. interference rejection) and with only positive frequency transfer function for the analytic signal construction. An example of filter following these properties is schemed on Figure 1. The resulting filter is a complex linear time-invariant filter whose transfer function is only defined in the positive frequencies. Let consider $\mathcal{H}$ such a filter with impulse response $h(t)$ and transfer function $H(f) = \int_{-\infty}^{+\infty} h(u)e^{-2i\pi fu}\, du$. If $\mathbf{X}$ is the filter input, the output is $\mathbf{Y} = \mathcal{H}[\mathbf{X}]$ defined by:

$$Y(t) = \int_{-\infty}^{+\infty} h(u)X(t-u)\, du \quad (4)$$

According to [12] and [18], an errorless reconstruction of $\mathbf{Y}$ can be performed if the spectral band $\mathcal{B}_X$ is divided into two foldings. Note, however, that this continuous-time filter is not implemented in practice but its transfer function is used to modify the expression of reconstruction formulas. Indeed, the reconstruction from PNS2 samples can be performed using two digital filters of complex gains $\eta_t(f) = \sum_{n\in\mathbb{Z}} a_n(t)e^{2i\pi fn}$ and $\psi_t(f) = \sum_{n\in\mathbb{Z}} b_n(t)e^{2i\pi fn}$. These filters take respectively $\mathbf{X}_a$ and $\mathbf{X}_b$ as inputs. This configuration amounts to orthogonal projections on the Hilbert spaces spanned respectively by $\mathbf{X}_a$ and $\mathbf{X}_b$ as detailed in [12], [18], [17]. Then, according to [18], exact reconstruction formula in mean-squared error sense can be derived in time domain:

$$\mathbf{Y}(t) = \sum_{n\in\mathbb{Z}} a_n(t)X(n+a) + \sum_{n\in\mathbb{Z}} b_n(t)X(n+b) \quad (5)$$

In practical applications, the infinite sums must be truncated since the signal reconstruction is performed from a finite length observation window. Let $2N$ be the length of the observation window, i.e. the number of available samples from each sequences. The truncated formula expresses as:

$$\tilde{\mathbf{Y}}(t) = \sum_{n=-N}^{N-1} a_n(t)X(n+a) + \sum_{n=-N}^{N-1} b_n(t)X(n+b) \quad (6)$$

The expressions of $a_n(t)$ and $b_n(t)$ are given in [18] for the exact reconstruction and for increasing convergence rates. In this paper, we derive these so-called interpolation functions for selective construction of the analytic signal for increasing convergence rates. In next section, filter transfer functions are specified through constraints on its bandwidth and regularity.

1. **Interference cancellation**: the filter must fit the signal sub-band ($H(f) = 0$ for $f \in \mathcal{B}_{\text{int}}$ and $H(f) = 1$ for $f \in \mathcal{B}_X$). This filter output corresponds to the interference-free signal of interest.

2. **Analytic signal construction**: the filter transfer function must be defined in the positive frequencies only and thus its impulse response is complex.

3. **Convergence rate improvement**: the interpolation functions $a_n(t)$ and $b_n(t)$ are derived using Fourier series expansions of the filter transfer function $H(f)$. From well-known properties of Fourier series expansions ([19]), higher convergence rate for the construction formula can be expected using transfer functions with increasing regularity.

The next section provides closed-form expressions of the selective analytic signal construction formulas with increasing convergence rate. Three different transfer functions and associated interpolations are provided.

3.2. Closed-form formulas for different convergence rates

3.2.1. Analytic rectangular filter

The classical formulas are obtained for a rectangular filter defined on $\mathcal{B}_{\mathcal{N}}(k)$. According to the method proposed in this paper, this filter bandwidth is first modified to fit the signal band in order to remove the interference. Moreover, only positive frequencies are kept to build the analytic signal. The resulting transfer function is given by:

$$H^{\mathcal{R}}_{F_1,F_2}(f) = \begin{cases} 1 & \text{for } f \in \mathcal{B}_X^+ \\ 0 & \text{elsewhere} \end{cases} \quad (7)$$

The derived associated interpolation functions $a_n(t)$ are:

$$a_n(t) = \frac{i\Delta e^{i[F_m(t-n-a)+2\pi k(a-b)]}\,\mathrm{sin}_c\left[\Delta(t-n-a)\right]}{\pi \sin\left[2\pi k(b-a)\right]} \quad (8)$$

with $\mathrm{sin}_c(x) = \frac{\sin(x)}{x}$. $b_n(t)$ is obtained by symmetry, changing a into b and reciprocally. As $H^{\mathcal{R}}_{F_1,F_2}(f)$ is not a continuous function of f at the points F_1 and F_2, the convergence rate is n^{-1} : $a_n(t) \underset{n\to\infty}{\propto} \frac{1}{n}$.

3.2.2. Analytic trapezoidal filter

Now in order to reach a convergence rate in n^{-2}, the rectangular filter is replaced by a trapezoidal filter defined on $\mathcal{B}_{\mathcal{N}}(k)$ by the transfer function:

$$H^{\mathcal{T}}_{F_1,F_2,B}(f) = \begin{cases} 1 & f \in \mathcal{B}_X^+ \\ \frac{-f+F_2}{B} + 1 & f \in (F_2, F_2+B) \\ \frac{f-F_1}{B} + 1 & f \in (F_1-B, F_1) \\ 0 & \text{elsewhere} \end{cases} \quad (9)$$

with the guard band parameter $B > 0$ such as $\frac{2k-1}{2} \leq F_1 - B$ and $F_2 + B \leq \frac{2k+1}{2}$. The interpolation functions $a_n(t)$ can be derived using filter series expansion:

$$a_n(t) = \frac{i\left(\Delta + \frac{B}{2}\right) e^{i[F_m(t-n-a)+2\pi k(a-b)]}}{\pi \sin\left[2\pi k(b-a)\right]} \times \mathrm{sin}_c\left[\frac{B}{2}(t-n-a)\right] \mathrm{sin}_c\left[(\Delta + \frac{B}{2})(t-n-a)\right] \quad (10)$$

$b_n(t)$ is obtained by symmetry. As $H^{\mathcal{T}}_{F_1,F_2,B}(f)$ is continuous over the interval $\mathcal{B}_{\mathcal{N}}(k)$ but not derivable in F_1-B, F_1, F_2 and F_2+B [19], the convergence rate is n^{-2}: $a_n(t) \underset{n\to\infty}{\propto} \frac{1}{n^2}$.

3.2.3. *Analytic raised cosine filter*

In order to reach a convergence rate in n^{-3}, an analytic raised cosine filter is defined on $\mathcal{B}_{\mathcal{N}}(k)$ by the transfer function:

$$H^{\mathcal{C}}_{F_1,F_2,B}(f) = \begin{cases} 1 & f \in \mathcal{B}^+_X \\ \cos^2\left[\frac{\pi}{2B}(f-F_2)\right] & f \in (F_2, F_2+B) \\ \cos^2\left[\frac{\pi}{2B}(f-F_1)\right] & f \in (F_1-B, F_1) \\ 0 & \text{elsewhere} \end{cases} \tag{11}$$

The interpolation functions $a_n(t)$ are defined as:

$$a_n(t) = \frac{i\pi\left(\Delta+\frac{B}{2}\right)e^{i[F_m(t-n-a)+2\pi k(a-b)]}}{B^2\sin\left[2\pi k(a-b)\right]\left[(t-n-a)^2-\left(\frac{\pi}{B}\right)^2\right]}\times \cos\left[\frac{B}{2}(t-n-a)\right]\mathrm{sin}_c\left[(\Delta+\frac{B}{2})(t-n-a)\right] \tag{12}$$

$b_n(t)$ is obtained by symmetry. As $H^{\mathcal{C}}_{F_1,F_2,B}(f)$ is derivable over $\mathcal{B}_{\mathcal{N}}(k)$ and its derivative is continuous over the same interval, the convergence rate is n^{-3}: $a_n(t) \underset{n\to\infty}{\propto} \frac{1}{n^3}$.

4. PERFORMANCE ANALYSIS

In the simulations, we consider a band-pass signal located in $\mathcal{B}_{\mathcal{N}}(3) = (-3.5, -2.5) \cup (2.5, 3.5)$. The signal of interest and the interference are modeled as white noises respectively in $\mathcal{B}_X = (-2.875, -2.625) \cup (2.625, 2.875)$ and $\mathcal{B}_{\text{int}} = (-3.375, -3.125) \cup (3.125, 3.375)$. The filter guard band parameter is $B = 0.125$. The received signal is sampled using PNS2 with $a = 0.3$ and $b = 0.777$. The formulas obtained with the three filters are compared through the estimation of the mean square error (MSE) between the estimated analytic signal (derived from a finite observation window) and the theoretical analytic signal of the signal of interest calculated using Hilbert transform (noted $\mathcal{H}_b$):

$$\text{MSE}(t) = \text{E}\left[\left|Y(t) - [X(t) + i\,\mathcal{H}_b[X](t)]\right|^2\right].$$

The MSE is estimated at the center of the observation window ($t = 0$), from $n_{it} = 1000$ signal runs. Figure 2 shows the influence of the parameter N (that sets the window size to $2N$). As expected, we retrieve the convergence rate for each filter. Figure 2 also compares the reconstruction on the entire band performed using formula (3) that clearly reconstructs the interference too, showing that our modified method allows the interference cancellation. Figure 3 displays the influence of the interference position now using a simple interference defined as a sine wave of frequency f_{int} (with random phase ditributed over $[0, 2\pi]$) and a fixed value of N ($N = 50$). The distance has no impact on the construction until the interference signal is located out of the filter band. Nevertheless we can note that more regular filters (trapezoidal and raised cosine) require to have a guard band which corresponds to the transition band of the filter (noted as B in the figure). For a rectangular filter, there is no guard band, allowing to have less binding conditions about the interference location.

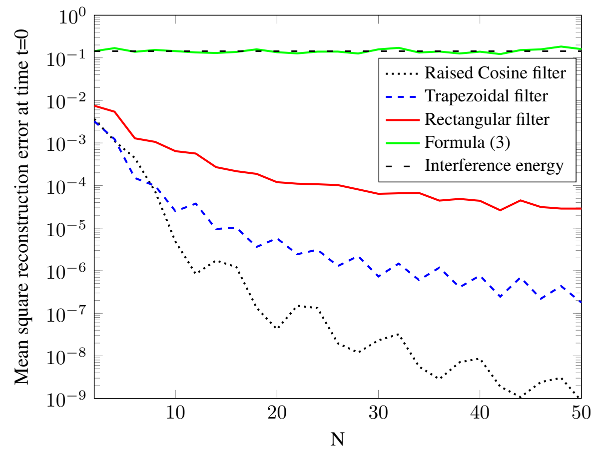

Fig. 2. Influence of the number of samples

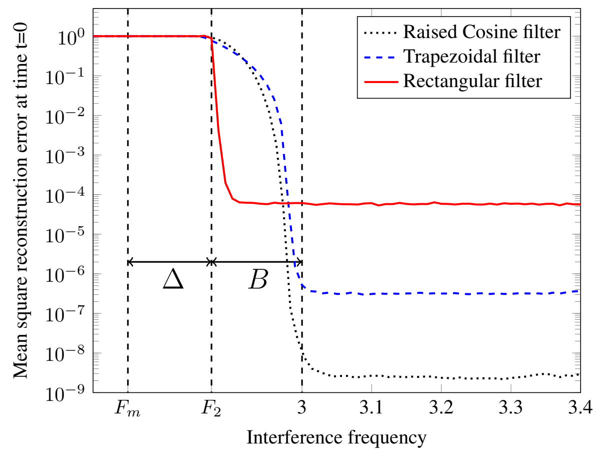

Fig. 3. Influence of the interference position ($N = 50$)

5. CONCLUSION AND FUTURE WORKS

This paper proposes a method that allows to perform simultaneously the digital reconstruction of a bandpass signal non-uniformly sampled by an operation of analog-to-digital conversion and a filtering operation upon this signal. This filtering operation has a double purpose as it allows to cancel an interference located outside the signal bandwidth and also to construct the analytic signal directly from the non-uniform samples of the onboard received signal. The formulas lead to an errorless construction. Based on the use of PNS2, the method allows to work directly and numerically on the band-pass signal without demodulation or transition into the frequency domain and with a low sampling frequency according to the Landau criterion. Coupled with specific filter designs that lead to fast convergence formulas, the proposed method can be useful in practical applications to overcome the problem of low number of available samples. The considered filter designs lead to convergence rate of respectively n^{-1} (rectangular), n^{-2} (trapezoidal) and n^{-3} (raised cosine). Even more regular filters can be constructed using piecewise polynomial transfer functions for instance. Such filters will lead to more complex formulas but also to higher convergence rates.

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