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# A NOVEL INTRINSIC CRAMÉR-RAO BOUND FOR EXACT GAUSSIAN DISTRIBUTION ON LIE GROUPS

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## ABSTRACT

In this communication, we propose a novel Cramér-Rao bound for matrix Lie groups, based on the exact modeling of Gaussian distributions on Lie groups (LG-ExCRB), different from those in the literature based on a simplifying approximation. To achieve this, we design a generic expression and develop new analytical formulas of the Fisher Information matrix. Then, closed-form expressions are given for two Lie groups of interest in engineering applications, $SO(3)$ and $SE(2)$. The proposed LG-ExCRB is validated numerically by comparison with the LG-CRB derived from the approximate modeling.

***Index Terms*—** Cramér-Rao bound, Gaussian distribution, Lie groups.

## 1. INTRODUCTION

Estimation on Lie groups (LGs) has become a major point of interest in a plethora of signal processing applications over the past few decades. Indeed, LGs are traditionally used to model observations/parameters constrained by geometric properties, such as rotation matrices, affine transformations, and covariance matrices [1, 2]. Examples in computer vision include motion averaging, structure from motion,[3] or image registration [4]. These tasks aim to estimate an affine transformation using the information provided by a camera. Also, in radar applications, tomographic synthetic aperture radar (SAR) imaging reconstructs a 3D environment by taking advantage of scattered matrix measurements modelled as symmetric positive definite (SPD) matrices [5], whereas in radar target tracking, LG structure allows to model the region of uncertainty of space debris measurements [6]. In all these applications, LG measurements can be generally fitted by a Gaussian distribution on LGs allowing to represent intrinsically the LG noise/uncertainty [7] and characterized by an LG-mean and an LG-covariance matrix. In this setting, similarly to the Euclidean case, the LG-mean can be estimated thanks to a maximum likelihood approach [8]. This is particularly relevant when the covariance matrix has low amplitude, i.e., small eigenvalues indicating low measurement variances. Indeed, in this case, the normalization term, depending on the curvature of the LG and on the LG-mean, is negligible. This approximation, leading to the concentrated Gaussian (LG-CGD) [9], is classically used in the literature typically to resolve intrinsic estimation problem on $SO(n)$ and $SE(n)$ [10, 11, 12]. In this context, it is therefore fundamental to determine the best achievable LG estimator in the sense of the mean square error to assess the performance of the maximum likelihood estimator, while taking into account their LG geometry. This performance limit is given by the Cramér-Rao bound on Lie groups (LG-CRB) [13], which is a bound on any unbiased estimator (in the LG sense) of the LG-MSE, for which closed-form expressions have been proposed and validated for the concentrated Gaussian model [14]. Nevertheless, the low amplitude covariance matrix assumption fails in diverse applications. As an example in computer vision involving motion estimation applications, two images from environments that are visually similar but slightly different can result in an inaccurately measured relative $SE(2)$ transformation [9] with high variance. Also, in a tracking context, inertial unit measures angular $SO(3)$ rotation using a gyrometer/odometer [15]. If it is poorly calibrated, the measurements can become highly inaccurate. Another example for space applications, star sensor can measure satellite $SO(3)$ orientation with high uncertainty primarily due to the stray light sources (Sun or Earth) [16].

To address this issue and assess the theoretical performance that can be achieved under high-noise variance LG measurements, we propose in this work to model LG data by using the exact form of the Gaussian distribution on LGs (LG-ExCGD). It follows that the main contribution of this paper is to derive a new Cramér-Rao bound on LGs, called LG-ExCRB, on the LG-mean that explicitly accounts for the true underlying distribution. To the best of our knowledge, this exact modeling has never been used in the literature. This is achieved by leveraging the general expression of the Fisher Information Matrix on Lie groups, which is developed by explicitly accounting for all terms of the LG-ExCGD. Unlike the LG-CRB derived under the concentrated model, the normalization term of the LG-ExCGD depends on the parameter to be estimated, thereby introducing new challenges in terms of differentiation. From the general formulation of the proposed LG-ExCRB, closed-forms of the Fisher information matrix (LG-FIM) are then obtained for the LG of interest previously



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introduced, $SO(3)$ and $SE(2)$. These expressions are computed by leveraging structural properties specific to each LG and requiring the computation of the derivative of the LG Jacobian. To our knowledge, they have never been established in the literature. The LG-ExCRB is implemented numerically and validated by comparison with the LG-MSE. More importantly, it is also compared with the LG-CRB derived under the concentrated model, in order to highlight the differences introduced by the exact modeling.

This communication is organized as follows: in Section II, we review background on statistics on LGs. In Section III, the proposed LG-ExCRB is introduced, and closed-form expressions are established. Finally, numerical experiments to validate the LG-ExCRB are provided in Section IV.

## 2. BACKGROUND ON ESTIMATION AND CRAMÉR-RAO BOUND ON LIE GROUPS

### 2.1. Properties of Lie Groups

A LG $G$ is a space that combines the properties of a group with those of a differentiable manifold. Its key property is that group operations are differentiable. Thus, it is possible to define a tangent space at each point $\mathbf{X} \in G$. This tangent space, $T_{\mathbf{X}}G$, is a vector space whose dimension $m$ matches that of the manifold. At the identity element $\mathbf{I}$, the tangent space $T_{\mathbf{I}}G$ is referred to as the *Lie algebra* and is denoted by $\mathfrak{g}$, which is in bijection with $\mathbb{R}^m$ as illustrated in Fig2.1. $G$ and $\mathfrak{g}$ are locally linked by the exponential group $\text{Exp}_G(.)$ and reciprocally by the logarithm map.

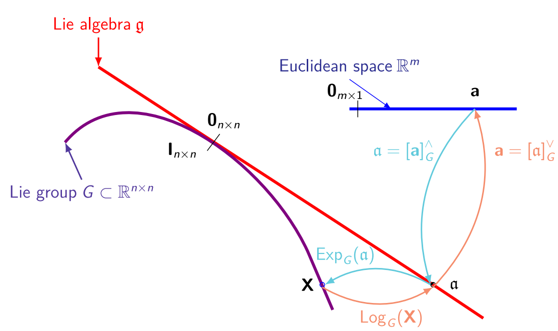


**Fig. 1**. Relation between $G$, $\mathfrak{g}$ and $\mathbb{R}^m$

### 2.2. Gaussian Distribution on Lie Groups

To obtain a distribution suitable for practical implementation, we can focus on the class of unimodular LGs, which are groups where the group measure is bi-invariant [2]. This property is respected by most LGs commonly used in the literature, such as $SO(n)$ and $SE(n)$. In this context, the maximum entropy distribution is achieved by the Gaussian Distribution on LGs (LG-ExGD) with LG-mean $\boldsymbol{\mu}$ and LG-covariance $\boldsymbol{\Sigma}$

$$p(\mathbf{Y}) = a(\mathbf{Y};\boldsymbol{\mu}) \exp\left(-\frac{1}{2}\|\boldsymbol{l}_G(\boldsymbol{\mu},\mathbf{Y})\|^2_{\boldsymbol{\Sigma}}\right), \quad \forall \mathbf{Y} \in G \quad (1)$$

$$\mathbf{Y} = \boldsymbol{\mu}\text{Exp}^\wedge_G(\boldsymbol{\epsilon}_i) \qquad \boldsymbol{\epsilon}_i \sim \mathcal{N}_{\mathbb{R}^m}(\mathbf{0},\boldsymbol{\Sigma}) \quad (2)$$

where $\|\cdot\|_{\boldsymbol{\Sigma}}$ represents the Mahalanobis distance and

$$a(\mathbf{Y};\boldsymbol{\mu}) \propto \frac{1}{\sqrt{|\boldsymbol{\Phi}\boldsymbol{\Sigma}\boldsymbol{\Phi}^\top|}}, \quad \boldsymbol{\Phi} = \boldsymbol{\Phi}_G(-\boldsymbol{l}_G(\boldsymbol{\mu},\mathbf{Y})), \quad (3)$$

$\boldsymbol{\Phi}_G(.) : \mathbb{R}^m \to \mathbb{R}^{m\times m}$ is the Jacobian matrix of $G$ and $\boldsymbol{l}_G(\mathbf{X},\mathbf{Y}) \triangleq \text{Log}^\vee_G\left(\mathbf{X}^{-1}\mathbf{Y}\right)$. Let us consider a set of observations $\{\mathbf{Z}_i\}_{i=1}^N$ distributed according to the LG-ExGD with unknown LG-mean $\mathbf{X}$ and known LG-covariance matrix $\boldsymbol{\Sigma}$. When $\boldsymbol{\Sigma}$ has small eigenvalues, the term $\boldsymbol{\Phi}$ is approached by $\mathbf{I}$ and we have the following so-called concentrated Gaussian distribution (LG-CGD)

$$p(\mathbf{Z}_i|\mathbf{X},\boldsymbol{\Sigma}) \propto \frac{1}{\sqrt{|\boldsymbol{\Sigma}|}} \exp\left(-\frac{1}{2}\|\boldsymbol{l}_G(\mathbf{X},\mathbf{Z}_i)\|^2_{\boldsymbol{\Sigma}}\right). \quad (4)$$

### 2.3. Cramér-Rao bound for Concentrated Gaussian distribution

**Definition 2.3.1.** Let us consider a set of observations $\{\mathbf{Z}_i\}_{i=1}^N$ distributed according to (4) $\forall i$ ($\mathbf{Z}_i \in G$ with dimension $m$). The LG-MSE between $\mathbf{X}$ and any unbiased estimator (in the LG sense) built from $\mathbf{Z}$ is [13]

$$\text{LG-MSE}(\mathbf{X},\widehat{\mathbf{X}}) = \mathbb{E}\left(\|\boldsymbol{l}_G(\mathbf{X},\widehat{\mathbf{X}})\|^2\right). \quad (5)$$

If the observations are mutually independent, the LG-MSE (5) is bounded by the LG-CRB given by

$$\mathbf{P}_{\text{LG-CRB}} = \mathcal{J}^{-1} \quad (6)$$

$$\mathcal{J} = \sum_{i=1}^N \mathbb{E}\left(\boldsymbol{\Psi}_G(\boldsymbol{l}_G(\mathbf{X},\mathbf{Z}_i))^\top \boldsymbol{\Sigma}^{-1}\,\boldsymbol{\Psi}_G(\boldsymbol{l}_G(\mathbf{X},\mathbf{Z}_i))\right) \quad (7)$$

with $\boldsymbol{\Psi}_G(.) \triangleq \boldsymbol{\Phi}_G(.)^{-1}$.

*Remark* 1. This formula of the LG-FIM $\mathcal{J}$ can be seen as a generalization of the well-known Slepian-Bangs formula for Euclidean observations [17] [18].

*Remark* 2. When $G$ is commutative, it implies that $\boldsymbol{\Psi}_G(.) = \mathbf{I}$ and $\mathbf{P}_{LG-CRB} = N^{-1}\boldsymbol{\Sigma}$.

## 3. DEVELOPMENT OF THE NEW CRAMÉR-RAO BOUND

In this section, we develop the proposed Cramér-Rao bound to address the problem described in Section 2. As previously, we assume independent observations $\{\mathbf{Z}_i\}_{i=1}^N$ but following

$$p(\mathbf{Z}_i|\mathbf{X},\boldsymbol{\Sigma}) = a(\mathbf{Z}_i;\mathbf{X}) \exp\left(-\frac{1}{2}\|\boldsymbol{l}_G(\mathbf{X},\mathbf{Z}_i)\|^2_{\boldsymbol{\Sigma}}\right) \quad (8)$$

### 3.1. Expression of the bound

**Theorem 1 (Generic expression of the LG-ExCRB).** *Under the model (7), the LG-MSE (5) for the LG-mean $\mathbf{X}$ is bounded by the exact Cramér-Rao bound (LG-ExCRB)*

$$\mathbf{P}_{\textit{LG-ExCRB}} = \mathcal{J}_{Ex}^{-1} \;\in \mathbb{R}^{m\times m} \quad (9)$$

$$\mathcal{J}_{Ex} = \sum_{i=1}^N \left(\mathcal{J}_1^i + \mathcal{J}_2^i + {\mathcal{J}_2^i}^\top + \mathcal{J}_3^i\right) \quad (10)$$



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$$\mathcal{J}_1^i = \mathbb{E}\left(\mathbf{t}_i\,\mathbf{t}_i^\top\right), \mathcal{J}_2^i = \mathbb{E}\left(\mathbf{t}_i\,\mathbf{q}_i^\top\right), \mathcal{J}_3^i = \mathbb{E}\left(\mathbf{q}_i\,\mathbf{q}_i^\top\right) \quad (11)$$

$$\mathbf{t}_i = \mathbf{\Psi}_G(\boldsymbol{l}_G(\mathbf{X},\mathbf{Z}_i))\,\frac{\partial log|\mathbf{\Phi}_G(-\mathbf{w})|}{\partial \mathbf{w}}\Big|_{\mathbf{w}=\boldsymbol{l}_G(\mathbf{X},\mathbf{Z}_i)} \quad (12)$$

$$\mathbf{q}_i = -\mathbf{\Psi}_G(\boldsymbol{l}_G(\mathbf{X},\mathbf{Z}_i))\boldsymbol{l}_G(\mathbf{X},\mathbf{Z}_i) \quad (13)$$

*Proof.* Let us recall that the LG-FIM has the following general expression, with $\boldsymbol{\delta}_1, \boldsymbol{\delta}_2 \in \mathbb{R}^m$

$$\mathcal{J} = \mathbb{E}\left[\frac{\partial \ell p(\mathbf{X},\boldsymbol{\delta}_1)}{\partial \boldsymbol{\delta}_1}\Big|_{\boldsymbol{\delta}_1=0}\left(\frac{\partial \ell p(\mathbf{X},\boldsymbol{\delta}_2)}{\partial \boldsymbol{\delta}_2}\right)^\top\Big|_{\boldsymbol{\delta}_2=0}\right] \quad (14)$$

where $\ell p(\mathbf{X},\boldsymbol{\delta}_j) = \log p(\mathbf{Z}_1,\ldots,\mathbf{Z}_N|\mathbf{X}\,\mathrm{Exp}_G^\wedge(\boldsymbol{\delta}_j))\forall j \in \{1,2\}$. By taking the logarithm of the likelihood of (8)

$$\ell p(\mathbf{X},\boldsymbol{\delta}_j)) = \underbrace{c}_{\in\mathbb{R}} - \frac{1}{2}\sum_{i=1}^{N}\left(\log|\mathbf{\Sigma}\,\mathbf{A}_i(\boldsymbol{\delta}_j)| + \left\|\tilde{\boldsymbol{l}}_G^i(\mathbf{X},\boldsymbol{\delta}_j)\right\|_{\mathbf{\Sigma}}^2\right) \quad (15)$$

where $\mathbf{A}_i(\boldsymbol{\delta}_j) \triangleq \mathbf{\Phi}_G(-\tilde{\boldsymbol{l}}_G^i(\mathbf{X},\boldsymbol{\delta}_j))\mathbf{\Phi}_G(-\tilde{\boldsymbol{l}}_G^i(\mathbf{X},\boldsymbol{\delta}_j))^\top$ and $\tilde{\boldsymbol{l}}_G^i(\mathbf{X},\boldsymbol{\delta}_j)) \triangleq \boldsymbol{l}_G(\mathbf{X}\mathrm{Exp}_G^\wedge(\boldsymbol{\delta}_j)\ ,\mathbf{Z}_i)$. Then

$$-\frac{\partial \ell p(\mathbf{X},\boldsymbol{\delta}_j)}{\partial \boldsymbol{\delta}_j} = \sum_{i=1}^{N}\underbrace{\frac{\partial}{\partial \boldsymbol{\delta}_j}\frac{1}{2}\log|\mathbf{A}_i(\boldsymbol{\delta}_j)|}_{\mathbf{f}_i^{(\boldsymbol{\delta}_j)}} + \underbrace{\frac{\partial}{\partial \boldsymbol{\delta}_j}\frac{1}{2}\left\|\tilde{\boldsymbol{l}}_G^i(\mathbf{X},\boldsymbol{\delta}_j)\right\|_{\mathbf{\Sigma}}^2}_{\mathbf{g}_i^{(\boldsymbol{\delta}_1)}} \quad (16)$$

Given the independence of the observations, this implies that

$$\mathcal{J} = \sum_{i=1}^{N}\mathbb{E}\left(\mathbf{f}_i^{(0)}\mathbf{f}_i^{(0)\top} + \mathbf{f}_i^{(0)}\mathbf{g}_i^{(0)\top} + \mathbf{g}_i^{(0)}\mathbf{f}_i^{(0)\top} + \mathbf{g}_i^{(0)}\mathbf{g}_i^{(0)\top}\right) \quad (17)$$

- *Computation of* $\mathbf{f}_i^{(\mathbf{0})} = \frac{\partial}{\partial \boldsymbol{\delta}_j}\frac{1}{2}\log|\,\mathbf{A}_i(\boldsymbol{\delta}_j)|\Big|_{\boldsymbol{\delta}_j=\mathbf{0}}$

  By applying $|\mathbf{B}\mathbf{B}^\top| = |\mathbf{B}|^2\ \forall\mathbf{B}$, and the chain rule

$$\mathbf{f}_i^{(0)} = -\frac{\partial}{\partial \boldsymbol{\delta}_j}\tilde{\boldsymbol{l}}_G^i(\mathbf{X},\boldsymbol{\delta}_j)\Big|_{\boldsymbol{\delta}_j=\mathbf{0}}\frac{\partial}{\partial \mathbf{w}}\log|\,\mathbf{\Phi}_G(-\mathbf{w})| \quad (18)$$

  computed at $\mathbf{w} = \boldsymbol{l}_G(\mathbf{X},\mathbf{Z}_i)$. By using BCH formula [19], we find $\frac{\partial}{\partial \boldsymbol{\delta}_j}\tilde{\boldsymbol{l}}_G^i(\mathbf{X},\boldsymbol{\delta}_j)\Big|_{\boldsymbol{\delta}_j=\mathbf{0}} = -\mathbf{\Psi}_G(\boldsymbol{l}_G(\mathbf{X},\mathbf{Z}_i))$ and (12).

- *Computation of* $\mathbf{g}_i^{(\mathbf{0})} = \frac{\partial}{\partial \boldsymbol{\delta}_j}\frac{1}{2}\left\|\tilde{\boldsymbol{l}}_G^i(\mathbf{X},\boldsymbol{\delta}_1)\right\|_{\mathbf{\Sigma}}^2\Big|_{\boldsymbol{\delta}_j=\mathbf{0}}$

  By using the chain rule

$$\frac{\partial}{\partial \boldsymbol{\delta}_j}\frac{1}{2}\left\|\tilde{\boldsymbol{l}}_G^i(\mathbf{X},\boldsymbol{\delta}_j)\right\|_{\mathbf{\Sigma}}^2\Big|_{\boldsymbol{\delta}_j=\mathbf{0}} = \frac{\partial}{\partial \boldsymbol{\delta}_j}\tilde{\boldsymbol{l}}_G^i(\mathbf{X},\boldsymbol{\delta}_j)\Big|_{\boldsymbol{\delta}_1=\mathbf{0}}\mathbf{\Sigma}^{-1}\boldsymbol{l}_G(\mathbf{X},\mathbf{Z}_i) \quad (19)$$

By applying a second time the BCH formula, we yield the expression in equation (13). Finally, by substituting the expressions of $\mathbf{f}_i^{(\mathbf{0})}$ and $\mathbf{g}_i^{(\mathbf{0})}$ in (17) we obtain the formula (10). □

### 3.2. Closed-form expressions for $SO(3)$ and $SE(2)$

In the expression of the LG-FIM (10) requires knowledge of both $\mathbf{t}_i$ and $\mathbf{q}_i$. While $\mathbf{q}_i$ is generally accessible—since it depends on known analytical expressions of $\mathbf{\Phi}_i$ and $\mathbf{\Psi}_i$ for most Lie groups of interest—$\mathbf{t}_i$ is typically not computable due to the inherent complexity of $\frac{\partial \log|\mathbf{\Phi}_G(-\mathbf{w})|}{\partial \mathbf{w}}|_{\mathbf{w}=\boldsymbol{l}_G(\mathbf{X},\mathbf{Z}_i)}$. To address this limitation, we propose to derive analytical formulas to deduce implementable expressions of the LG-FIM (and then LG-ExCRB) for the specific cases of $SO(3)$[1] and $SE(2)$[2]

**Corollary 1 (Expression for $SO(3)$).** *The LG-ExCRB is given for $\mathbf{X}\in SO(3)$ by (9), (10), (11) with*

$$\mathbf{t}_i = \mathbf{\Psi}_{SO(3)}(\boldsymbol{l}_{SO(3)}(\mathbf{X},\mathbf{Z}_i))\,\mathbf{d}(\boldsymbol{l}_{SO(3)}(\mathbf{X},\mathbf{Z}_i)) \quad (20)$$

$$\mathbf{d}(\mathbf{w}) = \left(cotan\,(\|\mathbf{w}\|) - \frac{2}{\|\mathbf{w}\|}\right)\frac{\mathbf{w}}{\|\mathbf{w}\|}\ \forall\mathbf{w}\in\mathbb{R}^3 \quad (21)$$

*Proof.* The aim is to analytically compute

$$\mathbf{t}_i = \mathbf{\Psi}_{SO(3)}(\boldsymbol{l}_{SO(3)}(\mathbf{X},\mathbf{Z}_i))\,\frac{\partial \log|\mathbf{\Phi}_{SO(3)}(-\mathbf{w})|}{\partial \mathbf{w}}\Big|_{\mathbf{w}=\boldsymbol{l}_{SO(3)}(\mathbf{X},\mathbf{Z}_i)} \quad (22)$$

by using the fact that [7]

$$\mathbf{\Phi}_{SO(3)}(\mathbf{w}) = \mathbf{I}_3 + a(\mathbf{w})\,\mathbf{\Omega} + b(\mathbf{w})\,\mathbf{\Omega}^2\ \ \forall\mathbf{w}\in\mathbb{R}^3. \quad (23)$$

with $a(\mathbf{w}) = \frac{1-\cos(\|\mathbf{w}\|)}{\|\mathbf{w}\|^2}$, $b(\mathbf{w}) = \frac{\|\mathbf{w}\|-\sin\|\mathbf{w}\|}{\|\mathbf{w}\|^3}$ and $\mathbf{\Omega} = [\mathbf{w}]_\times$. Its determinant can be computed by using spectral mapping theorem [20], and it ensues that each eigenvalue $\lambda_l^\phi$ of $\mathbf{\Phi}_G(\mathbf{w})$ depends on each eigenvalue $\lambda_l$ of $\mathbf{\Omega}, \forall l \in \{1,2,3\}$:

$$\lambda_l^\phi = 1 + a(\mathbf{w})\lambda_l + b(\mathbf{w})\lambda_l^2. \quad (24)$$

By straightforward computations, we obtain that $\{\lambda_l\}_{l=1}^3 = \{0, \pm i|\mathbf{w}|\}$ $\{\lambda_l^\phi\}_{l=1}^3 = \{1, (1-b(\mathbf{w})\|\mathbf{w}\|) \pm a(\mathbf{w})\|\mathbf{w}\|\}$ and $|\mathbf{\Phi}_{SO(3)}(\mathbf{w})| = (1-b(\mathbf{w}))^2 + a(\mathbf{w})$. By substituting expressions of $a(\mathbf{w})$ and $b(\mathbf{w})$, it results, after some tedious computations

$$|\mathbf{\Phi}_{SO(3)}(\mathbf{w})| = \frac{4\sin(\|\mathbf{w}\|^2)}{\|\mathbf{w}\|^2}. \quad (25)$$

By differentiating w.r.t.w, $\frac{\partial \log|\mathbf{\Phi}_{SO(3)}(-\mathbf{w})|}{\partial \mathbf{w}} = \mathbf{d}(\mathbf{w})$. □

**Corollary 2 (Expression for $SE(2)$).** *Let the residual term $\boldsymbol{l}_{SE(2)}(\mathbf{X},\mathbf{Z}_i) \triangleq [.,.,\theta^i] \in \mathbb{R}^3$. The LG-ExCRB is given by*

[1] $SO(3) = \{\mathbf{R}\in\mathbb{R}^{3\times3}|\mathbf{R}\mathbf{R}^\top = \mathbf{I}\}$ and $\mathfrak{so}(3) = \{|[\mathbf{w}]_\times|\mathbf{w}\in\mathbb{R}^3\}$.

[2] $SE(2) = \left\{\mathbf{T}\in\mathbb{R}^{3\times3}\,\middle|\,\mathbf{T} = \begin{bmatrix}\mathbf{R} & \mathbf{t}\\ \mathbf{0}_{1\times2} & 1\end{bmatrix},\ \mathbf{R}\in SO(2),\ \mathbf{t}\in\mathbb{R}^2\right\}$ and

$$\mathfrak{se}(2) = \left\{\mathfrak{a} = \begin{bmatrix}[w]_\times & \mathbf{u}\\ \mathbf{0} & 0\end{bmatrix}\middle|\, w\in\mathbb{R}, \mathbf{u}\in\mathbb{R}^2\right\},$$



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(9), (10), (11) *with*

$$\mathbf{t}_i = \alpha_i \, \boldsymbol{\psi}_3^i \quad \boldsymbol{\psi}_3^i : \textit{third column of } \boldsymbol{\Psi}_{SE(2)}(.) \tag{26}$$

$$\alpha_i = cotan\left(\frac{\theta^i}{2}\right) - \frac{2}{\theta^i} \tag{27}$$

*Proof.* In the same way as $SO(3)$, we want to compute $\mathbf{t}_i$ by using the expression of $\boldsymbol{\Phi}_{SE(2)}$ given by

$$\boldsymbol{\Phi}_{SE(2)}(\mathbf{w}) = \begin{bmatrix} \mathbf{J}(w_3) & \mathbf{s}(\mathbf{w}) \\ \mathbf{0}_{1\times 2} & 1 \end{bmatrix}, \mathbf{w} = [w_1, w_2, w_3] \in \mathbb{R}^3 \tag{28}$$

where $\mathbf{J}(w_3) = \frac{\sin(w_3)}{w_3}\mathbf{I}_{2\times 2} + \left[-\frac{1-\cos(w_3)}{w_3} \otimes \mathbf{I}_{2\times 1}\right]_\times$. $\mathbf{s}(\mathbf{w})$ depends non-trivially on $\mathbf{w}\cdot$ and $\otimes$ denotes the Kronecker product. Due to its sparse structure, we can directly compute its determinant

$$|\boldsymbol{\Phi}_{SE(2)}(\mathbf{w})| = \frac{2(1-\cos(w_3))}{w_3^2}, \tag{29}$$

which is similar to that of $SO(3)$. Then

$$\frac{\partial \log|\boldsymbol{\Phi}_{SE(2)}(-\mathbf{w})|}{\partial \mathbf{w}} = \left[0, 0, \text{cotan}\left(\frac{w_3}{2}\right) - \frac{2}{w_3}\right]^\top, \tag{30}$$

and $\mathbf{t}_i$ can be deduced by taking the product between $\boldsymbol{\psi}_{SE(2)}(.)$ and (30) computed at $\boldsymbol{l}_{SE(2)}(\mathbf{X}, \mathbf{Z}_i)$.
*Remark* 3. $\mathcal{J}_1^i = \mathbb{E}\left(\alpha_i^2 \boldsymbol{\psi}_3^i \, \boldsymbol{\psi}_3^{i\,\top}\right)$. □

*Remark* 4. The expectation $\mathbb{E}$ must be approximated via Monte Carlo simulation for different realizations of the observation noise. This is straightforward, as the Lie group Gaussian observations are generated according to the model $\mathbf{Z}_i = \mathbf{X}\,\text{Exp}_G^\wedge(\boldsymbol{\epsilon}_i)$ where $\boldsymbol{\epsilon}_i \sim \mathcal{N}_{\mathbb{R}^m}(\mathbf{0}, \boldsymbol{\Sigma})$.

## 4. SIMULATION RESULTS

To assess the proposed LG-ExCRB, we consider two simulations scenarios on $SO(3)$ and $SE(2)$. For both scenarios, the LG observations are generated according to (8). For $G = SO(3)$ the true LG-mean is set to $\mathbf{X} = \text{Exp}^\wedge_{SO(3)}\left(\left[\frac{\pi}{8}, \frac{\pi}{10}, \frac{\pi}{6}\right]\right)$ and $\boldsymbol{\Sigma} = \sigma^2 \mathbf{I}_{3\times 3}$. Regarding the scenario with $G = SE(2)$, we set $\mathbf{X} = \text{Exp}^\wedge_{SE(2)}\left([5, 5, \frac{\pi}{3}]\right)$ and $\boldsymbol{\Sigma} = \text{blkdiag}\left(\sigma_p^2 \otimes \mathbf{I}_2, \sigma_\theta^2\right)$.

In each scenario, the LG-MSE is assessed through two maximum likelihood estimators (MLE)—one $\widehat{\mathbf{X}}_{ML,c}$ that disregards the normalization term of the LG Gaussian, assuming it is negligible (LG-CGD), and another $\widehat{\mathbf{X}}_{ML,Ex}$ that incorporates it (LG-ExGD). Both are computed numerically by performing a Newton algorithm on LGs, and the LG-MSE for each estimator is evaluated over $Nr = 100$ Monte-Carlo realizations. These results are then compared to the trace of the proposed LG-ExCRB, computed according to corollaries (1) and (2), as well as to the LG-CRB defined in definition (2.3.1).

In the Fig. 2, we plot the results obtained for $SO(3)$ as a function of the variance $\sigma^2$ for $N = 100$. We observe that the two bounds are very close and tend to coincide as $\sigma^2$ decreases. However, the difference between them increases significantly as $\sigma^2$ becomes large. Similarly, in the Fig. 3, for $\sigma^2 = 0.8^2\,\text{rad}^2$ we observe that the two bounds exhibit a noticeable deviation as the number of observations $N$ increases. It demonstrates that for high variance and important number of measurements, an exact modeling of the LG-GD could theoretically yield a more accurate estimator. (The two MLE implemented are suboptimal for this value of $\sigma$). Fig. 4 illustrates the results obtained for $SE(2)$ as a function of $\sigma_p^2$, with $\sigma_\theta^2 = 0.01^2\,\text{rad}^2$ whereas Fig. 5 shows the influence of $\sigma_\theta$ with $\sigma_p^2 = 0.01^2\,\text{m}^2$. For both cases, $N$ is fixed at 100. As in the $SO(3)$ scenario, we observe that the two bounds tend to coincide as $\sigma_p^2$ or $\sigma_\theta^2$ decrease, and diverge for larger values, ensuring that the exact bound is less tight and underestimate less the optimal estimator. Note that the ML estimators tend to stabilize for high variance $\geq 0.5^2\,\text{rad}^2$, showing a certain robustness of the algorithms for poor rotation measurements.

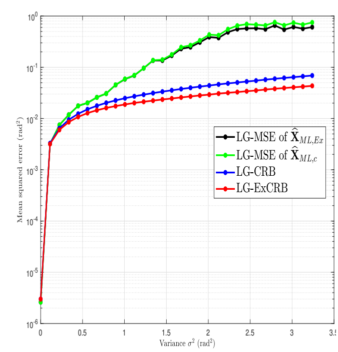


**Fig. 2.** Evolution of the LG-CRB and LG-ExCRB w.r.t. $\sigma_p^2$.

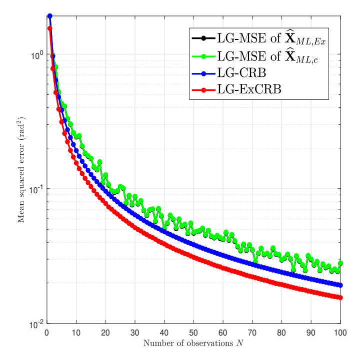


**Fig. 3.** Evolution of the LG-CRB and LG-ExCRB w.r.t. $N$.

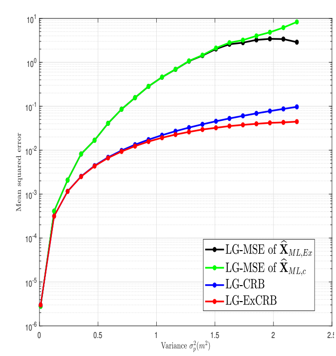


**Fig. 4.** Evolution of the LG-CRB and LG-ExCRB w.r.t. $\sigma_p^2$.

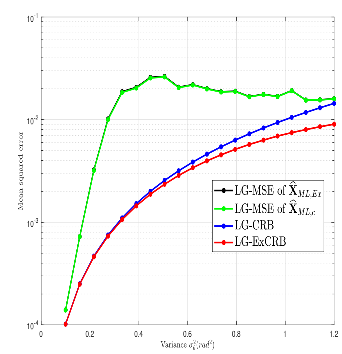


**Fig. 5.** Evolution of the LG-CRB and LG-ExCRB w.r.t. $\sigma_\theta^2$.

## 5. CONCLUSIONS AND PERSPECTIVES

In this communication, we propose a novel Cramér-Rao bound on Lie groups for exact Gaussian distributions, in-



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corporating curvature terms. This is achieved by deriving new analytical expressions of the LG-FIM. Simulation results demonstrate the effectiveness of the proposed bound on the LGs $SO(3)$ and $SE(2)$. Perspectives of this work would to design the LG-ExCRB in a Bayesian recursive context, crucial for on-line applications. Also, the high variance measurements could be modelled by robust non-Gaussian distributions and will yield to new analytical expressions of the LG-CRB.

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