Lie Group Bayesian Modeling of the von Mises Concentration Parameter

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***Abstract*—In this communication, we propose a new Bayesian framework to characterize the concentration parameter of the von Mises distribution. To achieve this, we equip this parameter with a Lie group structure. We design a Lie group (LG) estimator by incorporating prior information modeled by a Gaussian distribution on $\mathbb{R}^+$. This estimator is determined using a dedicated optimization algorithm on $\mathbb{R}^+$. The performance of this estimator is then evaluated by computing a new expression of the Bayesian Cramér-Rao bound on the Lie group (LG-BCRB) $\mathbb{R}^+$. The consistency between the proposed estimator and the LG-BCRB is validated through numerical simulations by comparing it with the Bayesian Mean Squared Error.**

***Index Terms*—Lie group, von Mises distribution, Bayesian estimation, Cramér-Rao bound.**

I. Introduction

The problem of processing angular data has emerged in many research areas within signal processing over the past decade. Such data are typically provided by various acquisition systems, including remote sensing systems (RADAR, LIDAR) and instrumental sensors such as gyrometers [1]–[3]. Classically, angle data lie on $]-\pi,\pi]$, making standard distributions like the Gaussian distribution inadequate for accounting for their periodicity [4], [5]. The von Mises distribution is one of the most popular distributions for fitting angular measurements and has demonstrated high modeling performance in various applications [4], [6], [7]. Additionally, it is well-suited for modeling signal phase observations in GNSS navigation systems within a Kalman filter architecture [8]–[10].

Starting with von Mises modeling whose probability density function is given by

$$p(\psi_i|\phi,\kappa) = \frac{1}{2\pi \mathrm{I}_0(\kappa)} \exp[\kappa \cos(\psi_i - \phi)], \tag{1}$$

($\mathrm{I}_0(\cdot)$: modified Bessel function) the first objective is to estimate the location parameter ϕ that provides a noise-free measurement. This estimation problem has been studied in the literature [11], [12]. However, estimating the second parameter κ, known as the concentration parameter and related to the variance of the distribution, is more challenging because it has no closed-form and its estimation is inaccurate for few observations [13], [14]. This problem is encountered in a plethory of applications as GNSS signal quality estimation [15], or thermal noise fluctuation characterization [14].

But also in sensor fusion applications where von Mises data is involved [16]. In environmental monitoring, the concentration estimation allows the assessment of wave measurement modeling ocean wave directions [17].

Although various solutions have been proposed to address the bias and the lack of accuracy, they remain empirical [18]. To overcome these issues, one solution is to change the space in which this parameter is expressed. A natural choice is to constraint κ in the set of positive values, which can be done by embedding it on the Lie group (LG) space $\mathbb{R}^+$ [19]. Lie groups are powerful tools that allow for the proper modeling of the geometrical properties of a parameter. They have demonstrated promising results across a plethora of applications [20]. Through this natural representation can be viewed as a geometric prior information, the precision of the estimation could be improved. Another solution is to define the problem within a Bayesian framework. Indeed, it is well-known that the mean squared error (MSE) of a Bayesian estimator depends only on its variance and is by definition unbiased.

In this article, we propose to combine these two solutions by developing an original Bayesian framework on LG and derive a new estimator of the concentration parameter of the von Mises distribution. It is important to emphasize that this is a preliminary work, and ultimately, the estimation of κ will be combined with that the one of ϕ. The first contribution of our work is to seek the estimator of κ by utilizing the LG structure of the positive real numbers, $\mathbb{R}^+$. The prior information is provided by a LG Gaussian distribution on $\mathbb{R}^+$, and the estimator is derived by defining a new LG cost function obtained from the posterior distribution of κ. Specifically, a Newton algorithm on the LG $\mathbb{R}^+$ is implemented [21]. The second contribution is to study the theoretical performance of this estimator by computing a LG Bayesian Cramér-Rao bound (LG-BCRB) on the concentration parameter. Based on the

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LG-BCRB developed in [22], a novel closed-form expression is derived for our problem. Contrary to some other LGs, it has the advantage of being provided without approximation due to the commutativity of $\mathbb{R}^+$. The consistency of the proposed estimator with respect to the LG-BCRB is validated through numerical experiments that study the impact of various parameters of the problem.
The communication is organized as follows: section II provides the necessary background on LGs, sections III and IV respectively develop the proposed LG estimator based on von Mises observations and the associated LG-BCRB, section V is dedicated to numerical simulations.

II. Backgrounds on Lie groups

In this section, we review background and useful Lie group tools for performing Bayesian estimation and deriving the Cramér-Rao bound.

A. Lie group: Definitions

A matrix Lie group $(G \subset \mathbb{R}^{n\times n}, \circledast)$ is a set of $n \times n$ matrices that forms a smooth manifold and a group under the operation $\circledast$. The tangent space at the identity matrix, called the Lie algebra $\mathfrak{g}$, is a vector space that serves as a local approximation of G. Elements of $\mathfrak{g}$ are linked to the Lie group through the exponential and logarithm maps $\exp_G : \mathfrak{g} \to G$ and $\log_G : G \to \mathfrak{g}$. If $\mathfrak{g}$ has dimension m, we can use bijections and their reciprocals $[.]^\wedge : \mathbb{R}^m \to \mathfrak{g}$ and $[.]^\vee : \mathfrak{g} \to \mathbb{R}^m$ to move between $\mathfrak{g}$ and $\mathbb{R}^m$. This structure is advantageous because it allows us to work with Euclidean vectors rather than matrices, which is more suitable from a numerical perspective. Fig.1 presents a schematic illustrating the relationship between the Lie group, Lie algebra, and Euclidean space.

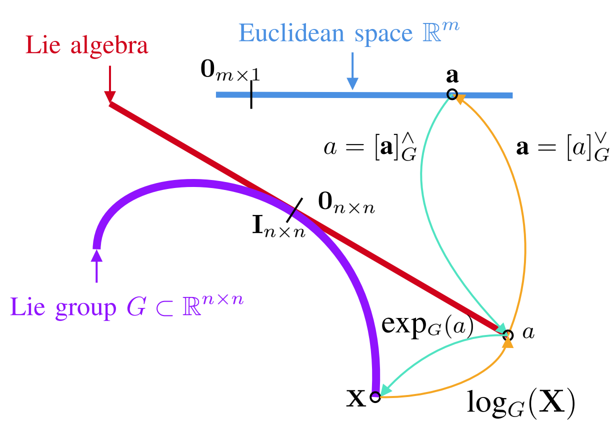

Fig. 1: Relation between Lie group, Lie algebra and Euclidean space.

The space $\mathbb{R}^+$ forms a commutative Lie group under the operation of classical multiplication, with the neutral element being 1. The logarithm and exponential maps for this group are given by the natural logarithm $\log(.)$ and the exponential function $\exp(.)$, respectively, both defined on $\mathbb{R}^+$ and $\mathbb{R}$.

B. Bayesian estimation on Lie groups

In this section, we review the Bayesian tools for Lie groups that are necessary in order to apply in our methodology. In the following, we propose to define all the concepts for a 1-dimensional Lie group G, i.e. $\log_G^\vee(.) \in \mathbb{R}$. We define the notation $\log_G^\vee(v)$ as the composition of the logarithm map $\log_G$ and $[.]^\vee$, such that $\log_G^\vee(v) = \log_G(v)$. Note that it can be easily generalized for any matrix Lie group for dimension greater than 1.

1) Gaussian distribution : To perform Bayesian inference on Lie groups (LGs), it is essential to define uncertainty i.e. probability distribution functions (pdfs). Here, we focus on the LG-Gaussian, generalizing the Euclidean Gaussian distribution. This distribution has several important properties, particularly the advantage of being easy to sample [22]. Let $\epsilon \in \mathbb{R}$ (isomorph to G) be a zero-mean Gaussian variable with variance σ^2, and let μ the mean defined on G. Then, the Gaussian distribution on Lie groups of $x = \mu \exp_G^\wedge(\epsilon)$ is given by[1]:

$$p(x) \simeq \frac{1}{\sqrt{(2\pi)\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}\log_G^\vee\left(\mu^{-1}x\right)^2\right). \quad (2)$$

In the case where G is commutative we can rewrite the latter distribution as:

$$p(x) = \frac{1}{\sqrt{(2\pi)\,\sigma^2}} \exp\left(-\frac{1}{2\,\sigma^2}\left(\log_G^\vee(x) - \log_G^\vee(\mu)\right)^2\right). \quad (3)$$

2) Posterior estimator: Let $x \in G$ be an unknown LG parameter estimated from a set of N independent Euclidean observations $\boldsymbol{\psi} = \{\psi_i\}_{i=1}^N$. We assume that x is *a priori* distributed according to a probability density function (pdf) $p(x)$ and that it is related to ψ through the posterior distribution $p(\boldsymbol{\psi}|x)$. A posterior estimator is basically given by the maximum a posterior (MAP)

$$\widehat{x} = \underset{x\in G}{\text{argmax}}\ p(x|\boldsymbol{\psi}) \quad (4)$$

While the likelihood depends on the nature of the sensor, it is conventional to assume that the prior is a Gaussian distribution on Lie groups as:

$$p(x|x_0, \sigma_0) = \frac{1}{\sqrt{(2\pi)\,\sigma_0^2}} \exp\left(-\frac{1}{2\,\sigma_0^2}\left(\log_G^\vee(x) - \log_G^\vee(x_0)\right)^2\right). \quad (5)$$

The advantage of using this distribution is that it allows for directly specifying a prior value on LG, with the associated uncertainty characterized by the variance σ_0^2. From a practical perspective, the latter represents the variance of the random variable $\log(\kappa)$. Thus, if the physical variance of κ, denoted by $\tilde{\sigma}_0^2$ is known, the corresponding value of σ_0^2 can be computed using the following transformation

$$\sigma_0^2 = \frac{\tilde{\sigma}_0^2}{\kappa_0}. \quad (6)$$

[1]The Gaussian can be also defined with $x = \exp_G^\wedge(\epsilon)\,\mu$

Due to the non-linear structure of the latter through the logarithm operator, the solution of (4) must be determined using numerical methods, primarily based on gradient descent techniques.

3) Bayesian Cramér-Rao bound on Lie groups: Finally, the accuracy of $\widehat{x}$ can be assessed by computing the Bayesian mean square error on LGs

$$\text{MSE}(x, \widehat{x}) = \int_{\psi} \int_{x} \log_G^{\vee}\left(x^{-1}\widehat{x}\right)^2 \, p(\psi \mid x)\, p(x)\, d\psi\, \mu(dx) \tag{7}$$

where μ denotes a Haar measure.

whose achievable minimum value is given by the Bayesian Cramér-Rao bound on Lie groups (LG-BCRB). Its expression is given by [22]

$$P = \mathbb{E}_{p(\boldsymbol{\psi},x)}\left(J(x,\widehat{x})\right) \mathcal{I}^{-1} \mathbb{E}_{p(\boldsymbol{\psi},x)}\left(J(x,\widehat{x})\right). \tag{8}$$

where $J(x,\widehat{x}) = \sum_{n=0}^{+\infty} \frac{B_n}{n!} ad_G(\log_G^{\vee}\left(x^{-1}\,\widehat{x}\right))^n$ such as B_n are the Bernoulli numbers, $ad_G(a)\, b = b\, a - a\, b \; \forall (a,b) \in \mathfrak{g}^2$ and $\mathcal{I}$ is the LG-Bayesian information Fisher matrix:

$$\mathcal{I} = -\mathbb{E}_{p(\boldsymbol{\psi},x)}\left(\frac{\partial^2 \log p\left(\psi, x \exp_G^{\wedge}(\epsilon_1)\ \exp_G^{\wedge}(\epsilon_2)\right)}{\partial \epsilon_1 \partial \epsilon_2} \Bigg|_{\epsilon_1,\epsilon_2=0} \right). \tag{9}$$

In the particular case where G is commutative, which we will interest in the following of the paper, then $J(x,\widehat{x}) = 1$ and

$$P = \mathcal{I}^{-1}. \tag{10}$$

III. Proposed von Mises Bayesian Estimation on Lie Groups

In this section, we introduce a novel Bayesian method for Lie groups, specifically tailored for observations following a von Mises distribution. We focus on the estimation of the concentration parameter: as it is related to the variance of the distribution, we propose to treat it as an intrinsic element of the Lie group $\mathbb{R}^+$. In addition, we consider a prior Gaussian distribution on the Lie group $\mathbb{R}^+$ and derive a new LG cost function for its estimation.

A. Problem statement

Let us assume a set of N independent angular observations $\{\psi_i\}_{i=1}^N$ following the von Mises distribution whose expression is given by 1. The likelihood function is given by:

$$p(\psi_1, ..., \psi_N | \phi, \kappa) = \prod_{i=1}^{N} p(\psi_i | \phi, \kappa). \tag{11}$$

In the following, we assume that the parameter ϕ is known (or already previously estimated) and we only focus on the estimation of κ. This choice is primarily made because an estimator of ϕ can be established very simply using estimation methods from the literature [23], [24]. Ultimately, both parameters will be jointly estimated. In the following, the dependence on ϕ is therefore eliminated from the likelihood.

The aim is to design a Bayesian estimator of κ on the Lie group $\mathbb{R}^+$. To accomplish this, it is essential to define a prior distribution that meets this constraint. We propose using the LG Gaussian distribution on $\mathbb{R}^+$, defined by

$$p(\kappa) = \frac{1}{\sqrt{2\pi\, \sigma_0^2}} \exp\left\{ -\frac{[\log(\kappa) - \log(\kappa_0)]^2}{2\sigma_0^2} \right\}, \tag{12}$$

where $\kappa_0 \in \mathbb{R}^+$ is the prior mean of κ with its variance denoted as σ_0^2.

B. Estimation problem

A conventional approach to this problem consists in identifying the value of κ that maximizes the posterior distribution of κ given by

$$p(\kappa | \psi_1, ..., \psi_N) \,\propto\, p(\psi_1, ..., \psi_N | \kappa)\, p(\kappa). \tag{13}$$

By using expressions (11) and (12), it ensues that

$$\hat{\kappa} = \underset{\kappa \in \mathbb{R}^+}{\text{argmax}} \prod_{i=1}^{N} \frac{1}{\text{I}_0(\kappa)} \exp\left[\kappa \cos(\psi_i - \phi)\right] \exp\left[-\frac{\tau(\kappa)^2}{2\sigma_0^2} \right], \tag{14}$$

where $\tau(\kappa) = \log(\kappa) - \log(\kappa_0)$. The latter equation can be reformulated in order to minimize the negative logarithm, thereby defining the estimation problem as the new following optimization problem on $\mathbb{R}^+$

$$\hat{\kappa} = \underset{\kappa \in \mathbb{R}^+}{\text{argmin}}\, h(\kappa), \tag{15}$$

where:

$$h(\kappa) = -\sum_{i=1}^{N} \left[\kappa \cos(\psi_i - \phi)\right] + N \log[\text{I}_0(\kappa)] + \frac{\tau(\kappa)^2}{2\sigma_0^2} \tag{16}$$

is the LG cost function. The latter does not admit any analytical solution due to the presence of the modified Bessel function. In order to address this issue, we propose to use a numerical approach thanks to a Newton algorithm on the Lie group $\mathbb{R}^+$ [21]. Generally designed for any matrix Lie group, the recursion on $\mathbb{R}^+$ is basically given, at each iteration l, by

$$\kappa^{l+1} = \kappa^l \exp(\delta^l), \tag{17}$$

where $\delta^l \in \mathbb{R}$ is

$$\delta^l = -H(\kappa^l)^{-1} \nabla h(\kappa^l), \tag{18}$$

and where $\nabla h(\kappa)$ and $H(\kappa)$ are respectively the LG gradient and Hessian of h computed as follows

$$\nabla h(\kappa) = \frac{\partial}{\partial \epsilon} h(\kappa \exp(\epsilon)) \Bigg|_{\epsilon=0}, \tag{19}$$

$$H(\kappa) = \frac{\partial}{\partial \epsilon_1 \partial \epsilon_2} h(\kappa \exp(\epsilon_1) \exp(\epsilon_2)) \Bigg|_{\epsilon_1=\epsilon_2=0}. \tag{20}$$

By using (16), and knowing that $\frac{\partial}{\partial \epsilon} \kappa \exp(\epsilon)\big|_{\epsilon=0} = \kappa$, we can demonstrate that

$$\nabla h(\kappa) = N\left[\kappa \frac{\text{I}_1(\kappa)}{\text{I}_0(\kappa)}\right] - \sum_{i=1}^{N} \left[\kappa \cos(\psi_i - \phi)\right] + \frac{\tau(\kappa)}{\sigma_0^2}, \tag{21}$$

$$H(\kappa) = N\left[\frac{\kappa^2}{\mathrm{I}_0(\kappa)}\left[\frac{\mathrm{I}_0(\kappa)+\mathrm{I}_2(\kappa)}{2}\right] - \kappa^2\frac{\mathrm{I}_1(\kappa)^2}{\mathrm{I}_0(\kappa)^2} + \kappa\frac{\mathrm{I}_1(\kappa)}{\mathrm{I}_0(\kappa)}\right] - \sum_{i=1}^{N}\left[\kappa\cos(\psi_i - \phi)\right] + \frac{1}{\sigma_0^2}. \quad (22)$$

In the Algorithm 1, we describe the Newton algorithm to estimate κ.

Algorithm 1 Newton-Based Bayesian Estimation of κ

Input: ϕ, σ_0, $\{\psi_i\}_{i=1}^N$, κ_0
Output: $\hat{\kappa}^l$
$\varepsilon_{min} \leftarrow 10^{-4}$, $\hat{\kappa}^l \leftarrow \kappa_0$, $\varepsilon \leftarrow \varepsilon_{min} + 1$,
while $\varepsilon > \varepsilon_{\text{min}}$ **do**
  $\hat{\kappa}^l_{\text{past}} \leftarrow \hat{\kappa}^l$
  Compute gradient $\nabla h(\kappa)$ (21)
  Compute Hessian $H(\kappa)$ (22)
  $\hat{\kappa}^l \leftarrow \hat{\kappa}^l_{\text{past}} \exp(\delta^l)$ (17)
  $\varepsilon \leftarrow |\hat{\kappa}^l_{\text{past}} - \hat{\kappa}^l|$
end while
return $\hat{\kappa}^l$

IV. Bayesian Cramér-Rao Bound on the LG $\mathbb{R}^+$

In order to assess the performance of the proposed Bayesian estimator on LG, it is essential to determine the theoretically best achievable value. Additionally, this error bound must account for the fact that the parameter belongs to $\mathbb{R}^+$. To address this, we derive the expression of the LG-BCRB based on the von Mises distribution and the prior distribution in this section.

A. Expression of the LG-BCRB for von Mises observations

Let us consider a set of N independent observations $\boldsymbol{\psi} = \{\psi_1, \ldots, \psi_N\}$ distributed according to (1). Furthermore, let us assume that κ is a priori distributed according to (12). The LG-BCRB on κ is given by:

$$P = \mathcal{I}^{-1} \quad (23)$$

$$\mathcal{I} = -\mathbb{E}_{p(\boldsymbol{\psi},\kappa)}\left(\frac{\partial^2 \log p\left(\boldsymbol{\psi}, \kappa \exp(\epsilon_1)\exp(\epsilon_2)\right)}{\partial\epsilon_1\partial\epsilon_2}\bigg|_{\epsilon_1,\epsilon_2=0}\right). \quad (24)$$

By using (1) and (12), it is expressed as

$$\mathcal{I} = \frac{1}{\sigma_0^2} + \mathbb{E}_{p(\kappa)}\left(N\left[\frac{\kappa^2}{\mathrm{I}_0(\kappa)}\left[\frac{\mathrm{I}_0(\kappa)+\mathrm{I}_2(\kappa)}{2}\right] - \kappa^2\frac{\mathrm{I}_1(\kappa)^2}{\mathrm{I}_0(\kappa)^2}\right]\right). \quad (25)$$

B. Demonstration

To achieve this, we first utilize the expression (9) to obtain the equation (24). Subsequently, we apply the conditional rule

$$\mathcal{I} = \mathcal{I}_l + \mathcal{I}_p. \quad (26)$$

with

$$\mathcal{I}_l = -\mathbb{E}_{p(\boldsymbol{\psi},\kappa)}\left(\sum_{i=1}^{N}\frac{\partial^2 \log p(\psi_i|\kappa\exp(\epsilon_1)\exp(\epsilon_2))}{\partial\epsilon_1\partial\epsilon_2}\bigg|_{\epsilon_1=\epsilon_2=0}\right) \quad (27)$$

$$\mathcal{I}_p = -\mathbb{E}_{p(\kappa)}\left(\frac{\partial^2 \log p(\kappa\exp(\epsilon_1)\exp(\epsilon_2))}{\partial\epsilon_1\partial\epsilon_2}\bigg|_{\epsilon_1=\epsilon_2=0}\right) \quad (28)$$

As $\sum_{i=1}^{N}\frac{\partial^2 \log p(\psi_i|\kappa\exp(\epsilon_1)\exp(\epsilon_2))}{\partial\epsilon_1\partial\epsilon_2}\bigg|_{\epsilon_1=\epsilon_2=0}$ is equal to the Hessian derived in (20), it results

$$\mathcal{I}_l = \mathbb{E}_{p(\psi,\kappa)}\left(N\left[\frac{\kappa^2}{\mathrm{I}_0(\kappa)}\left(\frac{\mathrm{I}_0(\kappa)+\mathrm{I}_2(\kappa)}{2}\right) - \kappa^2\frac{\mathrm{I}_1(\kappa)^2}{\mathrm{I}_0(\kappa)^2} + \kappa\frac{\mathrm{I}_1(\kappa)}{\mathrm{I}_0(\kappa)}\right] - \sum_{i=1}^{N}\left[\kappa\cos(\psi_i-\phi)\right]\right) \quad (29)$$

By using the fact that $\mathbb{E}_{p(\psi,\kappa)}(.) = \mathbb{E}_{p(\kappa)}\left(\mathbb{E}_{p(\psi|\kappa)}(.)\right)$ and that $\mathbb{E}_{p(\psi|\kappa)}(\cos(\psi_i - \phi)) = \frac{\mathrm{I}_1(\kappa)}{\mathrm{I}_0(\kappa)}$, we obtain

$$\mathcal{I}_l = \mathbb{E}_{p(\kappa)}\left[N\left[\frac{\kappa^2}{\mathrm{I}_0(\kappa)}\left[\frac{\mathrm{I}_0(\kappa)+\mathrm{I}_2(\kappa)}{2}\right] - \kappa^2\frac{\mathrm{I}_1(\kappa)^2}{\mathrm{I}_0(\kappa)^2}\right]\right]. \quad (30)$$

Regarding the term $\mathcal{I}_p$, we can readily demonstrate, by utilizing the expression (12) of $p(\kappa)$, that

$$\mathbb{E}_{p(\kappa)}\left[\frac{\partial^2 \log p(\kappa\exp(\epsilon_1)\exp(\epsilon_2))}{\partial\epsilon_1\partial\epsilon_2}\bigg|_{\boldsymbol{\epsilon_h}=0}\right] = -\frac{1}{\sigma_0^2} \quad (31)$$

Then, by combining (30) and (31), we acquire the equation (25).

V. Numerical Simulation

A. A-Evaluation of the LG estimator and LG-PCRB

In this subsection, we present the numerical simulations which were conducted in order to validate the proposed estimator and the LG-CRB. It was ran by generating N independent von Mises sequential measurements, and known constant ϕ angle of 10° and with a true value of $\kappa = 1$ to estimate. Then, we run algorithm (1) on $Nr = 10000$ Monte Carlo realizations of the measurements (ψ) and we compute the intrinsic Bayesian MSE of $\hat{\kappa}$ by Monte-Carlo approximation which is compared to the LG-BCRB. This process is repeated for several scenarios of simulations corresponding to three prior values of κ_0. For each of these values, we wish to study the evolution of both MSE and LG-CRB as a function of the number of measurements to validate their consistency while also studying the influence of the prior variance σ_0^2.

In Figs. 2, 3, 4 we show the evolution of the MSE and LG-BCRB respectively for $\kappa_0 = 2.2$, 3 and 4. First, it can be observed globally in all the figures that the MSE tends to converge to the LG-BCRB, although it does not completely reach its optimal value. Second, we note that regardless of the value of σ_0^2, the behavior of the MSE and LG-BCRB are coherent with respect to the value of N. Also, the LG-BCRB consistently remains below the MSE, which validates both the estimator and the expression of the proposed bound.

On the other hand, we remark also a consistency through the impact of σ_0^2. Indeed, for few observations ($N < 10$), when κ_0 is close to the true value, as in the Fig. 2, the error decreases when σ_0^2 becomes low. Conversely, when κ_0 deviates from κ, as shown in Figs. 3 and 4, the error increases as σ_0^2 decreases. These two behaviors are consistent suggesting that the algorithm seeks a compromise between the prior information and the measurements to find the true value of κ.

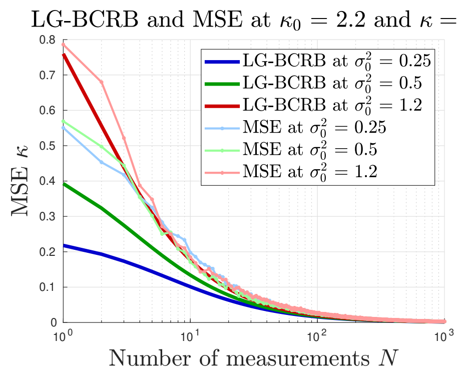

Fig. 2: Intrinsic MSE $\hat{\kappa}$ and associated LG-BCRB at $\kappa_0 = 2.2$.

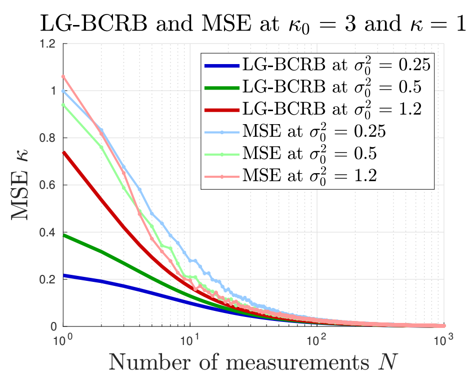

Fig. 3: Intrinsic MSE $\hat{\kappa}$ and associated LG-BCRB at $\kappa_0 = 3$.

B. Lie Group vs Classical approach

In this subsection, we analyze the the performance by using the intrinsic MSE of the LG approach against the Classic Euclidean approach denoted as "EU" (see Figs. 5, 6, 7). This approach consists in estimating kappa without considering its LG structure by using a Euclidean Newton algorithm with the same simulation conditions as the previous subsection. For $\kappa = 2.2$ (Fig. 5) where the a priori value is closer to the real κ, it can be observed that by having a high confidence in this scenario ($\sigma_0^2 = 0.25$), the LG approach outperforms the Euclidean in its entirety. For $\kappa_0 = 3$ (Fig. 6),

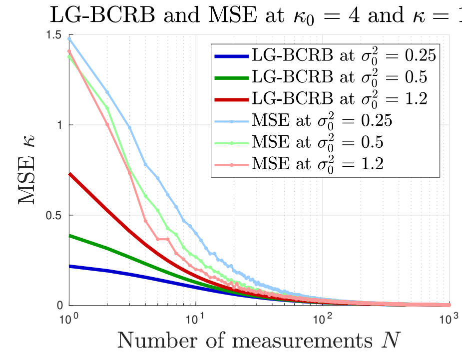

Fig. 4: Intrinsic MSE $\hat{\kappa}$ and associated LG-BCRB at $\kappa_0 = 4$.

we can observe how the performance of the Euclidean method degrades greatly compared to the LG method, which is more resilient to confidence at an a priori value farther from the true value. Finally, for the last scenario in Fig. 7 where $\kappa_0 = 4$, the farthest from the real value, we can observe that the LG approach outperforms the EU approach in all σ_0^2 conditions. It is relevant in the situation where the prior value is far from the true value.

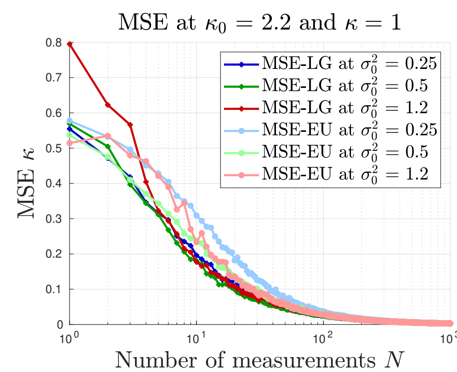

Fig. 5: Intrinsic MSE $\hat{\kappa}$ at $\kappa_0 = 2.2$.

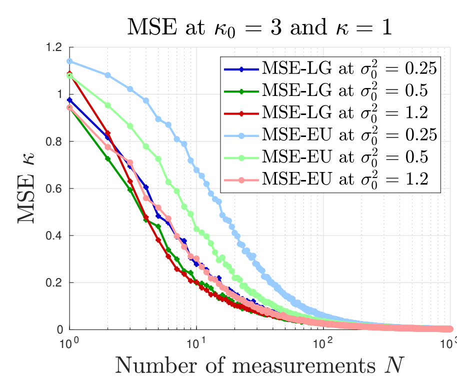

Fig. 6: Intrinsic MSE $\hat{\kappa}$ at $\kappa_0 = 3$.

VI. Conclusions

In this communication, we introduce a novel Bayesian approach for estimating the concentration parameter κ of the

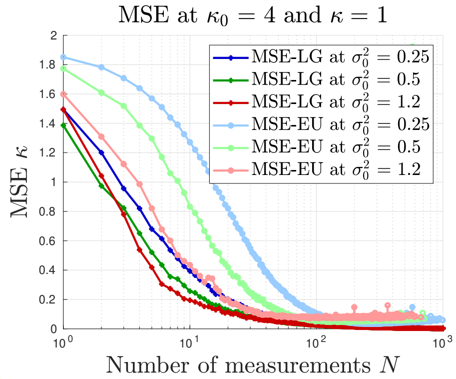

Fig. 7: Intrinsic MSE $\hat{\kappa}$ at $\kappa_0 = 4$.

von Mises distribution and its associated Bayesian Cramér-Rao bound. This method reformulates the estimation problem within a Lie group framework, providing an alternative to frequentist approaches. The derived estimation algorithm and LG-BCRB on Lie groups have been validated through numerical simulations, demonstrating the significant impact of prior values and their confidence on estimator performance when using a small number of measurements. This is particularly important in contexts like GNSS, where such conditions are found. Future work will focus on extending this problem to dynamic cases where the phase ϕ also needs to be estimated. A significant challenge will be to develop a recursive Bayesian estimator based on Bayesian filtering on Lie groups, as well as a recursive Bayesian Cramér-Rao bound.

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