Foldings of Periodic Nonuniform Samplings

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***Abstract*—Periodic Nonuniform Samplings of order N (PNSN) are interleavings of periodic samplings. For a base period T, simple algorithms can be used to reconstruct functions of spectrum included in an union Δ of N intervals δ_k of length $1/T$. In this paper we study the behavior of these algorithms when applied to any function. We prove that they result in N (or less) foldings on Δ, each of δ_k holding at most one folding.**

***keywords:* periodic nonuniform sampling, interpolation, foldings.**

I. Introduction

Let consider some real or complex function $g(t)$ with a (regular enough) Fourier transform $G(f)$ (the "spectrum" of $g(t)$)

$$g(t)=\int_{\Delta}e^{2i\pi ft}G(f)\,df \tag{1}$$

where Δ is the support of $G(f)$ (the set where $G(f)\neq 0$). The integral has a more general sense than a Riemann integral. For instance, $G(f)$ could be the sum of a continuous function with a linear combination of "Dirac functions". If $\widetilde{g}(t)$ is defined by

$$\widetilde{g}(t)=\sum_{n=-\infty}^{\infty}\operatorname{sinc}\left[\pi\left(\frac{t}{T}-n\right)\right]g(nT) \tag{2}$$

we know that $\widetilde{g}(t)=g(t)$ when $\Delta=\left]-\frac{1}{2T},\frac{1}{2T}\right[$ [1], [2], [3]. It is the simplest version of the "sampling formula" attributed to numerous scientists, for instance Cauchy, Shannon, Nyquist, Whittaker.... When the condition on Δ is not fulfilled, we have

$$\widetilde{g}(t)=\sum_{n=-\infty}^{\infty}h_n(t)\,e^{-2i\pi nt/T} \tag{3}$$

$$h_n(t)=\int_{(2n-1)/2T}^{(2n+1)/2T}e^{2i\pi ft}G(f)\,df \tag{4}$$

$h_n(t)$ is the component of $g(t)$ on the frequential interval $((2n-1)/2T,(2n+1)/2T)$, and the $h_n(t)\,e^{-2i\pi nt/T}$ are on $(-1/2T,1/2T)$. Whatever the "spectrum" $G(f)$ of $g(t)$, the sampling formula (2) "folds" $g(t)$ on $(-1/2T,1/2T)$.

Consequently, when we misuse the sampling formula (2), the result accumulates $h_0(t)$ (perhaps the interesting part of $g(t)$) with all other parts $h_n(t)$ shifted in frequency into $(-1/2T,1/2T)$.

This paper adresses the same problem in the framework of Periodic Nonuniform Samplings of order N (PNSN) which are interleavings of periodic samplings. We know interpolation formulas which are errorless in particular conditions. The question is: what happens when these formulas are misused? We will see that considerations above are generalized: the solution is a sum of N foldings matched to the spectrum of the analysed function. The problem has the same solutions when we replace the function $g(t)$ by the autocorrelation of a stationary process and $G(f)$ by the usual power spectrum.

II. Periodic Nonuniform Sampling of order N (PNSN)

1) A Periodic Nonuniform Sampling of order N (PNSN) is a sequence $\mathbf{t}$ of real numbers (the sampling times) in the form

$$\mathbf{t}=\{t_k+nT,k=1,..,N,n\in\mathbb{Z}\} \tag{5}$$

$0\leq t_1<t_2<....<t_N<T$. For $N=1$, the sequence is periodic. For $N>1$, $\mathbf{t}$ interleaves periodic sequences of same period T with delays t_k. Pioneering papers are attributed to J. L. Yen [4], and A. Kohlenberg [5] (for $N=2$). Results were diversely generalized [6], [7], [8]. Many papers assume that the t_k belong to particular sets (for instance $t_k\in(T/M)\,\mathbb{N}$ for some integer M). Here, the sequences are not subjected to such constraining hypotheses. Moreover, spectra of functions are not limited to the "baseband" case $\Delta=(-a,a)$.

Now, we consider the function $g(t)$ defined in (1) with (for some real μ and different α_k)

$$\Delta=\cup_{k=1}^{N}\left(\frac{\alpha_k}{T},\frac{\alpha_k+1}{T}\right),\alpha_k\in\mu+\mathbb{Z} \tag{6}$$

With respect to Δ, the sequence $\mathbf{t}$ verifies the "Landau condition": the length of Δ is equal to N/T and it is the mean number by unit time of sampling times (it is a generalization of the "Nyquist criterium" for sets which are finite unions of intervals [9]). Obviously, usual Δ can be approximated by or enclosed in sets like (6). We will disregard particular effects at the bounds of intervals of Δ for instance due to discontinuities of $G(f)$ at these points. Intervals are well suited for periodic samplings (PNS1). More general sets like (6) are matched to PNSN for well chosen N.

2) We consider functions $g(t)$ with Δ like (6), constituted by N intervals of length $1/T$. The set of samples

$$g(\mathbf{t})=\{g(t_k+nT),k=1,..,N,n\in\mathbb{Z}\}$$

is generally sufficient for an errorless reconstruction of $g(t)$, from the reconstruction of its "components" $g_l(t)$ defined as

$$g_l(t)=\int_{\alpha_l/T}^{(\alpha_l+1)/T}G(f)\,e^{2i\pi ft}df,\quad g(t)=\sum_{l=1}^{N}g_l(t) \tag{7}$$

In this case, the equation

$$\mathbf{Mg}(t)=\mathbf{H}(t) \tag{8}$$

$$\mathbf{M}=\left[e^{i\pi(2\alpha_l+1)t_k/T}\right],\mathbf{g}(t)=\left[g_l(t)\,e^{-i\pi(2\alpha_l+1)t/T}\right] \tag{9}$$

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TCSII.2021.3105870, IEEE Transactions on Circuits and Systems II: Express Briefs

$$\mathbf{H}(t)=\left[\sum_{n\in\mathbb{Z}}(-1)^n e^{-2i\pi n\mu}g(nT+t_k)\,\mathrm{sinc}\pi\left(\frac{t-t_k}{T}-n\right)\right] \tag{10}$$

is true (see [8], [10]). $\mathbf{g}(t),\mathbf{H}(t)$ are $(N,1)$ matrices. $\mathbf{M}$ is (N,N) and does not depend on t. Provided that $|\mathbf{M}|\neq 0$ ($|\mathbf{M}|$ is the determinant), the equation above has the unique solution

$$\mathbf{g}(t)=\mathbf{M}^{-1}\mathbf{H}(t) \tag{11}$$

(8) allows the determination of $g_l(t)$ for $l=1,..,N$ and $g(t)$ by (7). Results are linear combinations of sampling formulas which are the elements of $\mathbf{H}(t)$

III. Foldings

Sampling formulas (8) to (11) are true if $g(t)$ verifies some conditions as the integral representation (1), with (6) where $\alpha_k=m_k+\mu$ for some real μ and integers m_k. Otherwise, they define functions $\widetilde{g}_l(t),\widetilde{g}(t)$ different from $g_l(t),g(t)$.

In this more general situation, (9), (10) and (11) become

$$\widetilde{\mathbf{g}}(t)=\mathbf{M}^{-1}\mathbf{H}(t)\ \text{ with } \widetilde{\mathbf{g}}(t)=\left[\widetilde{g}_k(t)\,e^{-i\pi(2\alpha_k+1)t/T}\right] \tag{12}$$

where $\mathbf{M}^{-1}$ and $\mathbf{H}(t)$ are given by (8) and (it is a definition)

$$\widetilde{g}(t)=\sum_{l=1}^{N}\widetilde{g}_l(t) \tag{13}$$

A. The elementary case

Let assume that

$$g(t)=e^{2i\pi f_0 t},\, f_0\in\Delta_{\mu+m_0}=\left(\frac{\mu+m_0}{T},\frac{\mu+m_0+1}{T}\right) \tag{14}$$

for some f_0 which defines some integer m_0. We note $\widetilde{g}(t)=\widetilde{e}^{2i\pi f_0 t}$, $\widetilde{g}_l(t)=\widetilde{e}_l^{2i\pi f_0 t}$.

If $\mu+m_0=\alpha_k$ for some k, we have $\widetilde{g}(t)=\widetilde{g}_{m_0}(t)=e^{2i\pi f_0 t}$ and $\widetilde{g}_l(t)=0$ for $l\neq m_0$. Otherwise, let us introduce the complex functions $\beta_{l,m_0}(t), l=1,..,N$, such that ($\alpha_l=m_l+\mu$)

$$\begin{array}{rl}\widetilde{e}_l^{2i\pi f_0 t} & =\beta_{l,m_0}(t)\,e^{2i\pi t(f_0-(m_0-m_l)/T)}\\ \widetilde{e}^{2i\pi f_0 t} & =\sum_{l=1}^{N}\widetilde{e}_l^{2i\pi f_0 t}\end{array} \tag{15}$$

Actually, we prove that these functions are reduced to constants verifying (see Appendix)

$$\begin{bmatrix}\beta_{1,m_0}\\ \beta_{2,m_0}\\ ..\\ \beta_{N,m_0}\end{bmatrix}=\mathbf{M}^{-1}\begin{bmatrix}e^{i\pi(2m_0+2\mu+1)t_1/T}\\ e^{i\pi(2m_0+2\mu+1)t_2/T}\\ ..\\ e^{i\pi(2m_0+2\mu+1)t_N/T}\end{bmatrix} \tag{16}$$

Therefore, the solution of the problem is contained in the formula

$$\widetilde{e}^{2i\pi f_0 t}=\sum_{l=1}^{N}\beta_{l,m_0}e^{2i\pi t(f_0-(m_0-m_l)/T)} \tag{17}$$

where the β_{l,m_0} are computed from (16)

B. Consequences

Following properties are derived from (17) :

Property 1: when $f_0\in\Delta_{m_0+\mu}$, we have $f_0-(m_0-m_l)/T\in\Delta_{\alpha_l}$. So, $e^{2i\pi f_0 t}$ is folded on each Δ_{α_l} at a frequency which is different from f_0 by a multiple of $1/T$. It is possible that some lines disappear following the solutions of (16). Apart from some particular situations (for instance a periodic sampling), an undesirable monochromatic line at f_0 is split in N replicas in the sets Δ_{α_l} with (complex) amplitudes β_{l,m_0} defined by (16).

Property 2: the β_{l,m_0} do not depend on the place of the line f_0 in $\Delta_{\mu+m_0}$. Therefore, any $g(t)$ such that

$$g(t)=\int_{(m_0+\mu)/T}^{(m_0+\mu+1)/T}e^{2i\pi ft}G(f)\,df$$

will be folded as

$$\widetilde{g}(t)=g(t)\sum_{l=1}^{N}\beta_{l,m_0}e^{-2i\pi t(m_0-m_l)/T} \tag{18}$$

Property 3: but the β_{l,m_0} depend on m_0. So, in a more general case, we will have something like

$$\widetilde{g}(t)=\sum_{m}\left[\sum_{l=1}^{N}\beta_{l,m}e^{-2i\pi t(m-m_l)/T}\right]g_m(t) \tag{19}$$

where $g_m(t)$ is the component of $g(t)$ on $\Delta_{m+\mu}$.

Previous results are given in a complex framework where $e^{2i\pi f_0 t}$ is a unit spectral line at f_0, with $f_0\in\mathbb{R}$. When Δ is symmetric together with the "spectrum" $G(f)$, it is usual to consider only the positive part of the frequency axis. It is the case for real stationary processes where $G(f)$ is real and non-negative. In this framework, the unit spectral line is $\cos 2\pi f_0 t$, $f_0\geq 0$. Foldings are defined by

$$\widetilde{\cos}2\pi f_0 t=\frac{1}{2}\left(\widetilde{e}^{2i\pi f_0 t}+\widetilde{e}^{-2i\pi f_0 t}\right)$$

Examples 1 to 4 of section IV treat PNSN with $N=2,3,4$ for baseband or two-bands symmetric functions.

IV. Examples

Section III-B and formulas (18) and (19) prove that $\widetilde{e}^{2i\pi ft}$ and the set of $\beta_{l,m}$ define perfectly foldings in the case of PNSN fitted to (6). In the case of symmetric set Δ and real symmetric $g(t)$, foldings $\widetilde{g}(t)$ are real and even. In this situation, it is equivalent to calculate $\widetilde{\cos}2\pi f_0 t$ for positive values of f_0, as usual.

A. Example 1

We consider the case (PNS2 in baseband)

$$\Delta=(-1,1),\mathbf{t}=\{n,n+0.53,n\in\mathbb{Z}\}.$$

We have $\alpha_1=-1,\alpha_2=0,\mu=0$. Using (16), we find

$$\beta_{1,m}=-\frac{\sin 0.53\pi m}{\sin 0.53\pi}e^{0.53i\pi(m+1)}$$

$$\beta_{2,m}=\frac{\sin 0.53\pi(m+1)}{\sin 0.53\pi}e^{0.53i\pi m}.$$

We verify that we obtain an errorless reconstruction of $\cos 2\pi f_0 t$ when $f_0 \in \Delta_0$ $(m_0 = 0)$. In any other case, we have two lines at (positive) frequencies $m_0 + 1 - f_0$ and $f_0 - m_0$. Furthermore, the order of magnitude of the folded wave is that of the input. Figure 1 shows the positions and the amplitudes of lines of $\widetilde{\cos}2\pi f_0 t$ for $f_0 = 0.72, 1.72, 3.72$.

Actually, this PNS2 is a TI-ADC2 (based on 2 elementary ADC) with a "timing skew" equal to 0.03. If the latter value lacks accuracy, we will see 6 lines and not 5 in figure 1, which gives a method of estimation of timing skews (a parasite line will appear at the frequency 0.28 which will increase with the gap with 0.03). Furthermore, any signal $g(t)$ with spectral support $\Delta = (-\frac{1}{T}, \frac{1}{T})$ can be reconstituted without resampling, using (11) or specific formulas of PNS2 [11], [12]:

$$g(t) = \frac{-A_o(t)\sin\left(\pi\frac{t-\theta}{T}\right) + A_\theta(t)\sin\left(\pi\frac{t}{T}\right)}{\sin\left(\frac{\pi\theta}{T}\right)}$$

$$A_x(t) = \sum_{n\in\mathbb{Z}} (-1)^n \operatorname{sinc}[\pi(\frac{t-x}{T} - n)]g(nT + x) \qquad (20)$$

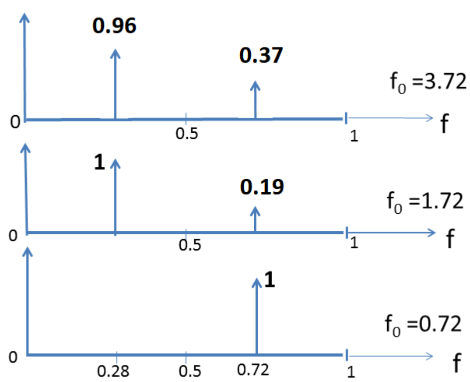

Fig. 1. Foldings for PNS2 in baseband (example 1), frequency-domain view.

B. Example 2

When $\Delta = (-\frac{1}{T}, \frac{1}{T})$, $N = 2$, $\theta/T = 1/2$ the periodic sampling allows a perfect reconstruction for functions in baseband. But the periodic sampling cannot give a good reconstruction when (for instance)

$$\Delta = \left(\frac{-3}{2T}, \frac{-1}{2T}\right) \cup \left(\frac{1}{2T}, \frac{3}{2T}\right), \mathbf{t} = \{nT, nT+\theta, n\in\mathbb{Z}\}$$

We are in the PNS2 two bands framework with (for instance)

$$\mu = -0.5, \alpha_1 = -1.5, \alpha_2 = 0.5$$

$$\beta_{1m} = -\frac{\sin\pi(m-1)\theta/T}{\sin 2\pi\theta/T} e^{i\pi\theta(m+1)/T}$$

$$\beta_{2m} = \frac{\sin\pi(m+1)\theta/T}{\sin 2\pi\theta/T} e^{i\pi\theta(m-1)/T}$$

The value $\theta/T = 1/2$ is inappropriate ($|\mathbf{M}| = 0$), and we know that a periodic sampling cannot be applied to this kind of spectrum. We retrieve the errorless formula for $f_0 \in \left(\frac{1}{2T}, \frac{3}{2T}\right)$, $m_0 = 1$, and two spectral lines elsewhere. At the opposite of example 1, $\widetilde{\cos}2\pi f_0 t$, as a function of θ/T, is not finite.

Figures 2 and 3 illustrate this example when $\theta = 0.41, T = 1$. Curves are done for $f_0 = 0.32, 1.32, 3.32$ corresponding to $m_0 = 0, 1, 3$. We find one line for $m_0 = 1$ (at 1.32) and two lines for $m_0 = 0$ and 3 (at 0.68 and 1.32). Examples 1 and 2 are based on PNS2, where more general situations can be treated [11], [12].

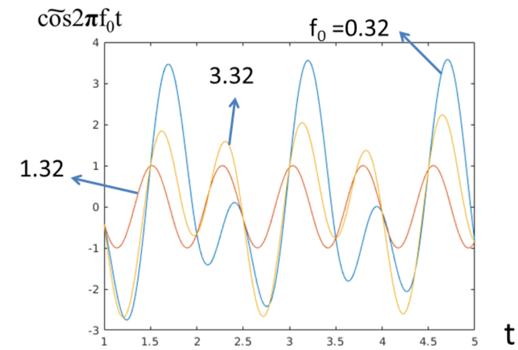

Fig. 2. Foldings for PNS2 in two-bands (example 2), time-domain view.

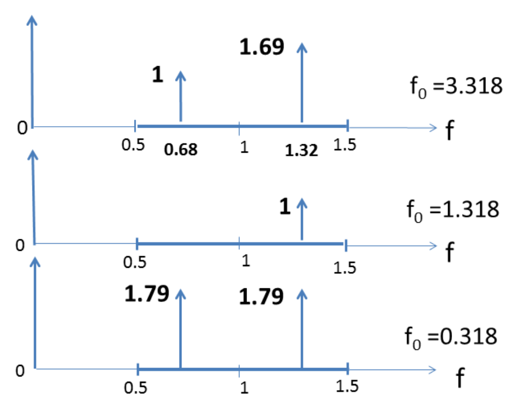

Fig. 3. Foldings for PNS2 in two-bands (example 2), frequency-domain view.

C. Example 3

We consider a PNS3 and a spectrum defined by

$$0 = t_1 < t_2 < t_3 < 1$$
$$\Delta = (-1.5, 1.5), T = 1$$

which corresponds to (for instance)

$$\mu = 0.5, \alpha_1 = -1.5, \alpha_2 = -0.5, \alpha_3 = 0.5$$

The case $m_0 = 1$ appears figure 4 and figure 5 with $f_0 = 1.57$, $t_1 = 0$, $t_2 = 0.05$ and $t_3 = 0.1$. The line at 1.57 is folded in lines at $f_0 - 1 = 0.57$, $-f_0 + 1 = 0.43$ and $-f_0 + 2 = 1.43$. Actually, the value 0.57 comes from $\widetilde{e}^{2i\pi f_0 t}$, and the others come from $\widetilde{e}^{-2i\pi f t}$

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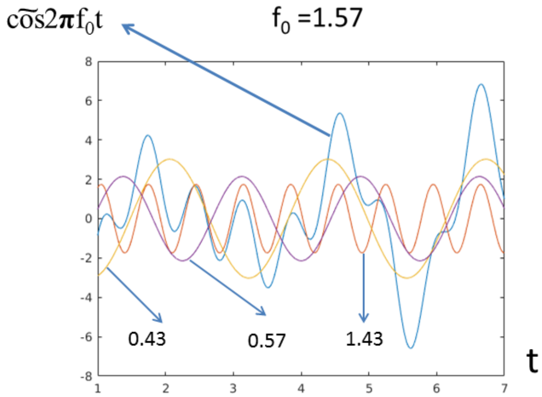

Fig. 4. Foldings for PNS3 in baseband (example 3), time-domain view.

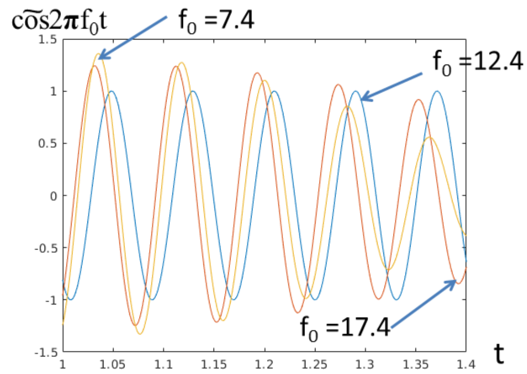

Fig. 6. Foldings for PNS4 in two-bands (example 4), time-domain view.

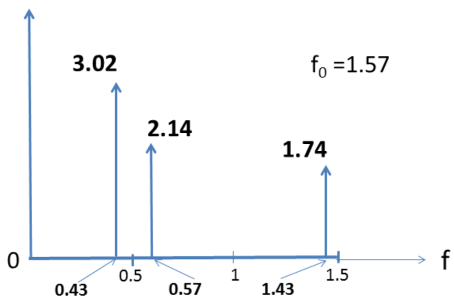

Fig. 5. Foldings for PNS3 in baseband (example 3), frequency-domain view.

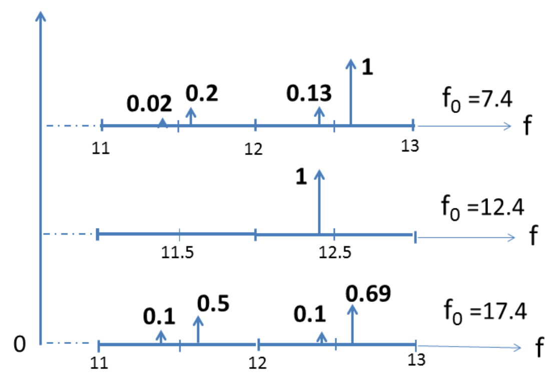

Fig. 7. Foldings for PNS4 in two-bands (example 4), frequency-domain view.

D. Example 4

We consider a PNS4 and a (two-bands) spectrum defined by

$$t_1 = 0, t_2 = 0.05, t_3 = 0.1, t_4 = 0.15$$
$$\alpha_1 = -13, \alpha_2 = -12, \alpha_3 = 11, \alpha_4 = 12, T = 1$$

As explained, we retrieve the (real) line at f_0 when $f_0 \in (11, 13)$. Otherwise, the line is folded in four places $y_1 ... y_4$ in $(11, 13)$: if $f_0 \in (m_0, m_0 + 1)$ for the positive integer m_0, we have

$$y_1 = -f_0 + m_0 + 13 \quad y_2 = -f_0 + m_0 + 12$$
$$y_3 = f_0 - m_0 + 11 \quad\quad y_4 = f_0 - m_0 + 12$$

Of course, two of them come from negative frequencies. Figures 6 and 7 are drawn for $f_0 = 7.4, 12.4, 17.4$, corresponding to $m_0 = 7, 12, 17$.

V. Conclusion

Periodic Non-uniform Sampling of order N (PNSN) are mixes of periodic samplings which appear in various contexts. For instance, in mechanical devices, PNS model Blade-Tip Timings due to detectors on the circumference of rotating engines [15], [18], [19].
More generally, irregular samplings can be approximated by PNS, with an accuracy which increases with N.

As an example, the Vostok Ice Core provides irregular datas (around 300) on the concentration of gases (CO_2,N_2O...) and of isotopes (D_2,O_{18}) during more than 400,000 years. These elements have a great importance in the study of wheather and climatic warming. Using PNSN allows a good reconstruction of the process and power spectra [8], [10].

A Time-Interleaved Analog-to-Digital Converter (TI-ADCN) is a parallel display of N elementary ADC which has to deliver a $\frac{T}{N}$ periodic sampling. It is a PNSN where t_k=$(k-1)T/N$. Actually, this equality is never satisfied, and the gaps are the "timing skews". Example 1 (IV-A) explains a method to recover the timing skew in the elementary case $N = 2$, and which is fitted to any N.

When $N = 4$, $\Delta = (-2/T, 2/T)$, we can have 3 timing skews $\mu_k = t_k - (k-1)T/4, k = 2, 3, 4$, $(t_1 = 0)$. μ_2 is estimated considering the PNS2 $\{nT, nT + t_2\}$, where t_2 is not well known but close to $T/4$. When $\mu_2 = 0$, formula 20 is true when applied to a function $g(t)$ with spectrum on $(-1/T, 1/T)$ and $\theta = T/4$. But the result is erroned when $\mu_2 \neq 0$. Small variations of the used parameter in the formula

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will allow to approach its true value. The operation can be repeated between the outputs of ADC 1 and 3, 1 and 4, and the results verified using ADC 2 and 4... Actually, taking $g(t) = \cos 2\pi f_0 t, 0 < f_0 < 1/T$ as test function generates a parasite spectral line, which disappears when a satisfactory value of the parameter is obtained. Once the t_k are better known, formula 20 allow to reconstruct functions without resampling.

Furthermore, reconstruction formulas (8) to (11) allow to overcome timing skews, without resorting to resamplings, whatever the order N.

This paper gives generalizations about the Periodic Nonuniform Sampling of order N (PNSN) plan. We consider spectra which are unions of N intervals

$$\left(\frac{\alpha_k}{T}, \frac{\alpha_k+1}{T}\right), k = 1, ..., N \text{ with } \alpha_k \in \mu + \mathbb{Z}$$

for some $\mu \in \mathbb{R}$. Such sets are able to approach many practical situations (we assume that the bounds shape have no incidence). Regarding the periodic case (PNS1), spectral lines are moved without change of amplitude. When $N > 1$, a spectral line is folded in each interval $\left(\frac{\alpha_k}{T}, \frac{\alpha_k+1}{T}\right)$, and we give the right places and the (complex) amplitudes.

Appendix A

With $g(t) = e^{2i\pi f_0 t}$, equation (12) becomes

$$\sum_{l=1}^{N} \widetilde{e}_l^{2i\pi f_0 t} e^{-i\pi(2\alpha_l+1)\frac{t-t_k}{T}} = \tag{21}$$

$$\sum_{n\in\mathbb{Z}} (-1)^n e^{-2i\pi n\mu+2i\pi f_0(nT+t_k)} \text{sinc}\pi \left(\frac{t-t_k}{T} - n\right)$$

for some $f_0 \in \Delta_{\mu+m_0}$ $(m_0 \in \mathbb{Z})$. When $\mu + m_0 = \alpha_k$ for some k, we have $\widetilde{e}_l^{2i\pi f_0 t} = e_l^{2i\pi f_0 t}$ for all l. Elsewhere, $f_0 - (m_0 + \mu - \alpha_l)/T \in \Delta_{\alpha_l}$, which allows to write (8) under the form $(k, l = 1, .., N)$

$$e^{2i\pi(t-t_k)\left(f_0-\left(m_0+\mu-\frac{1}{2}\right)/T\right)} = \tag{22}$$

$$\sum_{n\in\mathbb{Z}} (-1)^n e^{-2i\pi n\mu+2i\pi f_0 nT} \text{sinc}\pi \left(\frac{t-t_k}{T} - n\right)$$

Using $(21), (22)$, yields

$$\sum_{q=1}^{N} \left[\widetilde{e}_l^{2i\pi f_0 t} e^{-2i\pi\frac{t}{T}(f_0-m_0+m_q)}\right] e^{2i\pi\frac{t_k}{T}\left(m_q+\mu+\frac{1}{2}\right)} = \tag{23}$$

$$e^{2i\pi\frac{t_k}{T}\left(m_0+\mu+\frac{1}{2}\right)}.$$

wich is equivalent to the matricial equality

$$\mathbf{B} = \mathbf{M}^{-1}\mathbf{K} \tag{24}$$

where $\mathbf{M} = \left[e^{2i\pi\frac{t_k}{T}\left(m_q+\mu+\frac{1}{2}\right)}\right]$, $\mathbf{K} = \left[e^{2i\pi\frac{t_k}{T}\left(m_0+\mu+\frac{1}{2}\right)}\right], \mathbf{B} = \left[\widetilde{e}_l^{2i\pi f_0 t} e^{-2i\pi\frac{t}{T}(f_0-m_0+m_q)}\right]$.
$\mathbf{B}$ is independent of t, and f_0 appears through m_0,which proves (16) with (17).

References

[1] J. R. Higgins, *Sampling Theory in Fourier and Signal Analysis,* Oxford Sc. Pub. (1996).

[2] A. Papoulis, *Signal Analysis,* Mc-Graw Hill (1977).

[3] A. J. Jerri, *The Shannon sampling theorem. Its various extensions and applications. A tutorial review,* Proc. IEEE **65** (11) (1977) 1565-1596.

[4] J. L. Yen, *On Nonuniform Sampling of Bandwidth-Limited Signals,* IRE Trans. on Circ. Th. **CT-3** (12) (1956) 251-257.

[5] A. Kohlenberg, *Exact interpolation of band-limited functions,* J. Appl. Physics **24** (12) (1953) 1432-1436.

[6] Y. Lin, P. P. Vaidyanathan, *Periodic Nonuniform Sampling of Bandpass Signals,* IEEE Trans. on Circuits and Systems-II, **45** (3) (3-1998) 340-351.

[7] R. Venkataramani, Y. Bresler, *Sampling Theorems for Uniform and Periodic Nonuniform MIMO Sampling of Multiband Signals,* IEEE Trans. on Signal Proc. **51** (12) (2003) 3152-3163.

[8] B. Lacaze, *Reconstruction from Periodic Nonstationary Sampling (PNS) without Resampling*, Samp. Th. in Sign. and Im. Proc. **16** (2017) 1-21.

[9] H. J. Landau, *Sampling, Data Transmission, and the Nyquist Rate,* Proc. of the IEEE **35** (10) (10-1967) 1701-1706.

[10] D. Bonacci, B. Lacaze, *New CO_2 concentration predictions and spectral estimation applied to the Vostok ice core,* IEEE. Trans. on Geoscience and Remote Sampling **56** (1) (2018) 145-151.

[11] B. Lacaze, *About Bi-periodic Samplings,* Samp. Th. Sign. Im. Proc. **8** (3) (2009) 287-306.

[12] B. Lacaze, *Equivalent Circuits for the PNS2 Sampling Scheme,* IEEE Circuits and Systems **57** (11) (2010) 2904-2914.

[13] J. A. Vernhes et all, *Adaptative Estimation and Compensation of the Time Delay in a PNS,* SAMPTA Washington DC 5-2015.

[14] J. A. Vernhes et all, *Blind estimation of unknown time delay in PNS: application to desynchronized TI-ADC,* ICASSP Shangaï 3-2016.

[15] B. Lacaze, *A Blade-Tip Timing Method Based on Periodic Nonuniform Sampling of order 2,* arXiv: 1711.06135v1[physics.data-an] $11-14-$ 2017.

[16] B. Reyes, M. Hueda, A. Pola, *Design and Experimental Evaluation of a TI-ADC Calibration Algorithm for Application in High-Speed Communication Systems,* IEEE Trans. on Circ. and Sys. 1, **64** (5) (2017) 1019-1030.

[17] J. Gao et all, *An adaptative calibration technique of timing skew mismatch in time interleaved analog-to-digital converters,* Rev. Sci. Instrum. **90**, 025102 (2019).

[18] S. Heath, M. Imregun, *A review of analysis techniques for blade tip-timing measurements*, AMSE congress, Orlando 1997.

[19] Zheng Hu et all, *A Non-Uniformly Under-Sampled Blade Tip-Timing Signal Reconstruction Method for Blade Vibration Monitoring,* Sensors **15** (2015) 2419-2437.