Misspecified Time-Delay and Doppler Estimation Over High Dynamics Non-Gaussian Scenarios

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***Abstract*—This article focuses on the study of time-delay and Doppler estimation under high dynamic non-Gaussian scenarios. We aim at analysing the Mean Squared Error (MSE) performance of a misspecified receiver architecture which deliberately simplifies the signal model by neglecting the acceleration parameter and assumes the noise process as complex normal distributed. Specifically, we derive the pseudo-true parameters by minimazing the Kullback-Leibler (KL) divergence between the true and assumed models and the related Misspecified Cramér-Rao Bound (MCRB) will be provided in closed form. Theoretical derivations are validated via Monte Carlo simulations showing the asymptotic efficiency of the Misspecified Maximum Likelihood Estimator (MMLE). One remarkable outcome of this study is that the lack of knowledge of the true statistical noise model does not lead to asymptotic performance degradation in the estimation of the parameters of interest.**

***Index Terms*—Complex elliptically symetric distribution, Misspecified Cramér-Rao bound, time-delay and Doppler estimation, band-limited signals.**

I. Introduction

The estimation of deterministic signal parameters plays a crucial role in various applications, such as navigation, radar, or communications [1]–[10]. This problem has garnered significant attention over the past decades and has become an integral part of modern signal processing. Generally, noisy signal observations are assumed to be sampled from a well-specified family of parametric distributions, i.e. the *statistical model*. However, in certain scenarios, the true parametric model can be deliberately misspecified to simplify the estimation of the parameters of interest [11]–[14]. This might entail choosing to estimate fewer parameters than those actually influencing the signal dynamics. For instance, many applications related with the GNSS and Radar systems assume that the effects of receiver and/or target acceleration are negligible [15]. In other words, instead of attempting to jointly estimate time-delay, Doppler, and acceleration, the focus is solely on estimating time-delay and Doppler. In previous works, concise analytical expressions of the Cramér-Rao Bound (CRB) have been introduced for GNSS and radar systems receiver architectures. Specifically, the closed forms of the CRB, along with the investigation of the MSE performance of the related Maximum Likelihood Estimator (MSE), have been provided in [16], [17] for time-delay and Doppler and in [18], [19] for time-delay, Doppler, and acceleration, assuming a band-limited signal. Moreover, the assessment of performance limits in terms of MSE for a misspecified receiver architecture, where the acceleration is not considered in the parameter space, was first explored in [15]. Specifically, in [15], it is shown that under certain acceleration ranges, it is convenient to implement the misspecified maximum likelihood estimator (MMLE) instead of the well-specified MLE. Moreover, building upon the Slepian-Bangs formula derived in [20], a new compact closed-form expression of the MCRB [13] for the time-delay and Doppler parameters estimation under high dynamic scenarios was derived. However, the main assumption adopted in [15] is that both the true and assumed noise distributions are centred complex-normal distributions. Nonetheless, real-world scenarios often deviates from the Gaussian assumption, with the true data process being a heavy-tail and non-Gaussian one. These assumptions have been introduced for the first time in [21], for low dynamics scenarios. Thus, in this article, we assume that the true noise process is characterised by a Complex Elliptically Symmetric (CES)-distribution and we present the two following contributions: *i*) we derive the pseudo-true parameters by minimizing the Kullback-Leibler (KL) divergence between the true and assumed observation models and we prove that the pseudo-true parameters of interest equate the ones obtained by considering a complex-normal distribution and *ii*) we derive a closed-form MCRB expression for the parameter of interest under the assumption of band-limited signal and we prove that this expression is similar to the MCRB considering a true complex-normal distribution. Finally, we validate the theoretical outcomes via Monte Carlo simulations for one representative CES distribution: the Generalised Gaussian *GG* one.

II. True and misspecfied Signal model

A. True Signal Model

Let us consider that a band-limited signal $a(t)$, with bandwidth B, is transmitted over a carrier frequency f_c ($\lambda_c = c/f_c$, $\omega_c = 2\pi f_c$) from a transmitter T at position $\boldsymbol{P}_T(t)$ to a receiver R at position $\boldsymbol{P}_R(t)$. The radial displacement between transmitter and receiver $p_{TR}(t) = \|\mathbf{p}_T(t) - \mathbf{p}_R(t)\|$ is proportional to the signal time-delay, which is in-turn affected by the relative motion described by the relative velocity $\mathbf{v} = \mathbf{v}_T - \mathbf{v}_R$ and relative acceleration $\mathbf{a} = \mathbf{a}_T - \mathbf{a}_R$ (where $\mathbf{p}_T(t) = \mathbf{p}_T(0) + \mathbf{v}_T t + \frac{1}{2}\mathbf{a}_T t^2$ and $\mathbf{p}_R(t) = \mathbf{p}_R(0) + \mathbf{v}_R t + \frac{1}{2}\mathbf{a}_R t^2$) between both transmitter and receiver. This distance is used in the ranging equation for tracking of the target $p_{TR}(t;\boldsymbol{\eta}) = c\tau_{true}(t;\bar{\boldsymbol{\eta}})$, where c is the speed of light and $\tau_{true}(t;\bar{\boldsymbol{\eta}})$ represents the delay as a function of time and the *true* parameters $\bar{\boldsymbol{\eta}} = [\bar{\tau}, \bar{b}, \bar{d}]^\top \in \mathbb{R}^3$ that describe the displacement dynamics. Thus, the equation which describes the line of sight distance travelled by the transmitted signal can be approximated, up to the second order, as [22]:

$$p_{TR}(t;\bar{\boldsymbol{\eta}}) = \|\mathbf{p}_T\left(t - \tau_{true}(t;\bar{\boldsymbol{\eta}})\right) - \mathbf{p}_R\left(t\right)\| = c\tau_{true}(t;\bar{\boldsymbol{\eta}}) \simeq \left\|\mathbf{p}_T(0) - \mathbf{p}_R(0) - \mathbf{v}t - \frac{1}{2}\mathbf{a}t^2\right\|, \quad (1)$$

with $\tau_{true}\left(t;\bar{\boldsymbol{\eta}}\right) \simeq \bar{\tau} + \bar{b}t + \bar{d}t^2, \bar{\tau} = \frac{\|\mathbf{p}_T(0)-\mathbf{p}_R(0)\|}{c}, \bar{b} = \frac{\|-\mathbf{v}\|}{c}, \bar{d} = \frac{\|-\mathbf{a}\|}{2c}$. The received "noise-free" signal at the output of the Hilbert filter can be expressed as [15], [19]

$$s\left(t;\bar{\boldsymbol{\eta}}\right) = \bar{\alpha}a\left(t-\bar{\tau}\right)e^{-j2\pi f_c\left(\bar{b}(t-\bar{\tau})+\bar{d}(t-\bar{\tau})^2\right)} \quad (2)$$

with $\bar{\alpha} = \bar{\rho}e^{j\bar{\Phi}}$. It is worth pointing out that in (2) the narrowband assumption is adopted, i.e. the Doppler and acceleration parameters does not have a direct impact on the the baseband signal samples. In order to take into account the random channel effects, a noise term is added leading to:

$$x(t) = s\left(t;\bar{\boldsymbol{\eta}}\right) + n(t), \quad (3)$$

where $n(t)$ is defined as complex non-Gaussian, wide-sense stationary, continuous-time noise process. The discrete signal model is built from $N = N_1 - N_2 + 1$ samples at $T_s = 1/F_s = 1/B$,

$$\mathbf{x} = \bar{\alpha}\boldsymbol{\mu}(\bar{\boldsymbol{\eta}}) + \mathbf{n} = \bar{\rho}e^{j\bar{\Phi}}\boldsymbol{\mu}(\bar{\boldsymbol{\eta}}) + \mathbf{n}, \quad (4)$$

with $\mathbf{x} = \left(\ldots, x\left(kT_s\right), \ldots\right)^\top$, $N_1 \leq k \leq N_2$ signal samples. The samples of the non-Gaussian noise process $n(t)$, i.e. $\mathbf{n} = \left(\ldots, n\left(kT_s\right), \ldots\right)^\top$ are assumed to be zero-mean, Complex Elliptically Symmetric distributed $n(kT_s) \sim CES(0, \bar{\sigma}_n^2, g)$ with variance $\bar{\sigma}_n^2$ and unspecified density generator g [23]. Furthermore, we assume that the noise samples are *independent* and *identically distributed* (i.i.d.). Moreover, from eq. (3), an explicit expression for each entry $\boldsymbol{\mu}_k(\bar{\boldsymbol{\eta}})$ of $\boldsymbol{\mu}(\bar{\boldsymbol{\eta}})$ can be obtained as:

$$\mu_k(\bar{\boldsymbol{\eta}}) = a(kT_s - \bar{\tau})e^{-j2\pi f_c\left(\bar{b}(kT_s-\bar{\tau})+\bar{d}(kT_s-\bar{\tau})^2\right)}. \quad (5)$$

For further reference, the true "data-generating" parameters can be gathered in a vector $\bar{\boldsymbol{\epsilon}}^\top = \left(\bar{\sigma}_n^2, \bar{\rho}, \bar{\Phi}, \bar{\boldsymbol{\eta}}^\top\right) \in \Gamma \subset \mathbb{R}^+ \times \mathbb{R}^+ \times [0, 2\pi] \times \mathbb{R}^3$.

It follows immediately from the above definition of the considered signal model, that the *true* probability density function (pdf) of the non-Gaussian random vector $\mathbf{x} \in \mathbb{C}^N$ in (4) is given by $\mathbf{x} \sim p_{\bar{\epsilon}} \triangleq p_{\mathbf{x}}(\mathbf{x};\bar{\boldsymbol{\epsilon}}) = \Pi_{k=N_1}^{N_2} p_{x_k}(x_k, \bar{\boldsymbol{\epsilon}})$, with $p_{x_k}(x_k, \boldsymbol{\epsilon}) = CES(\bar{\alpha}\boldsymbol{\mu}_k(\bar{\boldsymbol{\eta}}), \bar{\sigma}_n^2, g)$. From the Stochastic Representation Theorem [23, Theo. 3], we have:

$$x_k =_d \bar{\alpha}\mu_k(\bar{\boldsymbol{\eta}}) + \sqrt{\mathcal{Q}}\bar{\sigma}_n u_k, \quad (6)$$

where $u_k \in \mathbb{C}$ is a complex uni-variate random variable uniformly distributed on $\mathbb{C}S \triangleq \{u \in \mathbb{C} \| u| = 1\}$, i.e. $u_k \sim U(\mathbb{C}S)$. The *second order modular variate* $\mathcal{Q}$ is a positive random variable, independent from u_k with pdf $p_{\mathcal{Q}}(q) = \delta_g^{-1}g(q)$, where $\delta_g \triangleq \int_0^\infty g(q)dq$ is a normalizing constant (see [23, Eq. (19)]). To avoid the well-known scale ambiguity between $\bar{\sigma}_n^2$ and g, we impose that $E\{\mathcal{Q}\} = 1$. Note that, this constraint allows us to consider $\bar{\sigma}_n^2$ as the *statistical power* P of the data x_k, (see the discussion in [23, Sec. III.C]), since from (6), we have that:

$$P \triangleq E\{|x_k - \bar{\alpha}\mu_k(\bar{\boldsymbol{\eta}})|^2\} = E\{\mathcal{Q}\}E\{|u_k|^2\}\bar{\sigma}_n^2 = \bar{\sigma}_n^2, \quad (7)$$

since $E\{|u_k|^2\} = 1$ [23, Lemma 1].

B. Misspecified Signal Model

Unfortunately, in standard receivers it is not possible to implement acceleration-aware estimators within the signal model due to its complexity. In addition, as a second simplifing assumption, the noise vector $\mathbf{n}$ in eq. (4) is usually considered as a zero-mean, complex Gaussian vector with diagonal covariance matrix and statistical power σ_n^2. Clearly, the two above-mentioned simplifications lead to a *model misspecification*. In particular, since the acceleration will not be considered, one can define a reduced (and misspecified) version of the continuous signal model in eq. (3) as:

$$s_{miss}(t;\boldsymbol{\gamma}) = \alpha a(t-\tau)e^{-j2\pi f_c b(t-\tau)}. \quad (8)$$

As for the true signal, we can build the misspecified discrete model from N samples at T_s as:

$$s_{miss}(kT_s;\boldsymbol{\gamma}) \triangleq \alpha\kappa_k(\boldsymbol{\gamma}) = \alpha a(kT_s - \tau)e^{-j2\pi f_c b(kT_s-\tau)} \quad (9)$$

The misspecified signal parameters can be cast in a vector $\boldsymbol{\gamma} = (\tau, b)^\top \in \mathbb{R}^2$ and the complete set of unknown misspecifed parameters is $\boldsymbol{\phi}^\top = (\sigma_n^2, \rho, \Phi, \boldsymbol{\gamma}^\top) = (\sigma_n^2, \boldsymbol{\theta}^\top) \in \Psi \subset \mathbb{R}^+ \times \mathbb{R}^+ \times [0, 2\pi] \times \mathbb{R}^2$. It is important to note that, since the acceleration is not considered in the misspecified estimation problem, the true and the assumed parameter spaces, i.e. Γ and Ψ respectively, are different, and specifically $\Gamma = \Psi \times \mathbb{R}$. Moreover, note that, while the true (uniquely defined) parameter vector is indicated as $\bar{\boldsymbol{\epsilon}} \in \Gamma$, a generic misspecified parameter vector in Ψ is denoted as $\boldsymbol{\phi} \in \Psi$. By collecting the previous definitions, the Gaussian-based, "acceleration-unaware" statistical model for the observation vector in (4) can be expressed as:

$$\mathcal{F}_{\boldsymbol{\phi}} \triangleq \left\{f_{\boldsymbol{\phi}} | f_{\mathbf{x}}(\mathbf{x};\boldsymbol{\phi}) = \mathcal{CN}(\alpha\boldsymbol{\kappa}(\boldsymbol{\gamma}), \sigma_n^2\boldsymbol{I}_N); \boldsymbol{\phi} \in \Psi\right\}. \quad (10)$$

Since, in general, $p_{\bar{\epsilon}} \notin \mathcal{F}_{\boldsymbol{\phi}}$, no estimator can estimate the true parameter vector $\bar{\boldsymbol{\epsilon}} \in \Gamma$. The best that we can do is to estimate the so called *pseudo-true* parameter vector $\boldsymbol{\phi}_0 \in \Psi$ [13], [14], that is the vector that minimizes the Kullback-Leibler divergence (KLD), i.e. $D(p_{\bar{\epsilon}} \| f_{\boldsymbol{\phi}}) = E_{p_{\bar{\epsilon}}}\left[\ln p_{\mathbf{x}}(\mathbf{x};\bar{\boldsymbol{\epsilon}}) - \ln f_{\mathbf{x}}(\mathbf{x};\boldsymbol{\phi})\right]$ between any $f_{\boldsymbol{\phi}} \in \mathcal{F}_{\boldsymbol{\phi}}$ and the true pdf $p_{\bar{\epsilon}}$. Moreover, since the assumed misspecified parameter space Ψ is nested in the true one, i.e. $\Gamma = \Psi \times \mathbb{R}$, the best we can expect is that the pseudo-true parameter vector $\boldsymbol{\phi}_0^\top = (\sigma_0^2, \boldsymbol{\theta}_0^\top) \in \Psi$ equates the subvector $\bar{\boldsymbol{\phi}} \triangleq (\bar{\sigma}_n^2, \bar{\rho}, \bar{\Phi}, \bar{\tau}, \bar{b}) = (\bar{\sigma}_n^2, \bar{\boldsymbol{\theta}}^\top)^\top \in \Psi$ of the true parameter vector $\bar{\boldsymbol{\epsilon}}^\top = (\bar{\boldsymbol{\phi}}^\top, \bar{d}) \in \Gamma$.

III. THE PSEUDO-TRUE PARAMETER VECTOR

The pseudo-true parameter vector $\boldsymbol{\phi}_0 \in \Psi$ is defined as

$$\boldsymbol{\phi}_0 \triangleq \arg\min_{\boldsymbol{\phi}\in\Psi} D(p_{\bar{\boldsymbol{\epsilon}}}||f_{\boldsymbol{\phi}}) = \arg\min_{\boldsymbol{\phi}\in\Psi} E_{p_{\bar{\boldsymbol{\epsilon}}}}\left[-\ln f_{\boldsymbol{\phi}}\right] \text{ where,} \tag{11}$$

$$\begin{aligned} E_{p_{\bar{\boldsymbol{\epsilon}}}}\left[-\ln f_{\boldsymbol{\phi}}\right] &= E_{p_{\bar{\boldsymbol{\epsilon}}}}\left[-\ln f_{\mathbf{x}}(\mathbf{x};\boldsymbol{\phi})\right],\ \mathbf{x}\sim p_{\bar{\boldsymbol{\epsilon}}} \\ &= N\ln(\pi\sigma_n^2) + E_{p_{\bar{\boldsymbol{\epsilon}}}}\left[\frac{||\mathbf{x}-\alpha\boldsymbol{\kappa}(\boldsymbol{\gamma}))||^2}{\sigma_n^2}\right]. \end{aligned} \tag{12}$$

Let us start by minimising with respect to (w.r.t.) the subvector $\boldsymbol{\theta} \in \Xi \subset \mathbb{R}^+ \times [0, 2\pi] \times \mathbb{R}^2$ of $\boldsymbol{\phi}\top = (\sigma_n^2, \boldsymbol{\theta}_0) \in \Psi$. This minimisation leads to the following expression that does not depend on σ_n^2:

$$\begin{aligned} \boldsymbol{\theta}_0 &= \arg\min_{\boldsymbol{\theta}\in\Xi}\left\{E_{p_{\bar{\boldsymbol{\epsilon}}}}\left[-\ln f_{\mathbf{x}}(\mathbf{x};\boldsymbol{\phi})\right]\right\},\ \mathbf{x}\sim p_{\bar{\boldsymbol{\epsilon}}} \\ &= \arg\min_{\boldsymbol{\theta}\in\Xi}\left\{E_{p_{\bar{\boldsymbol{\epsilon}}}}\left[\|\mathbf{x}-\alpha\boldsymbol{\kappa}(\boldsymbol{\gamma})\|^2\right]\right\} \\ &= \arg\min_{\boldsymbol{\theta}\in\Xi}\left\{\|\bar{\alpha}\boldsymbol{\mu}(\bar{\boldsymbol{\eta}}) - \alpha\boldsymbol{\kappa}(\boldsymbol{\gamma})\|^2\right\}. \end{aligned} \tag{13}$$

By posing to zero the following derivative, the pseudo-true noise variance σ_0^2 can be evaluated as:

$$\begin{aligned} E_{p_{\bar{\boldsymbol{\epsilon}}}}\left[\frac{\partial}{\partial\sigma_n^2}\ln f_{\sigma_n^2,\boldsymbol{\theta}_0}\right] &= \frac{N}{\sigma_n^2} - \frac{E_{p_{\bar{\boldsymbol{\epsilon}}}}\left[\|\mathbf{x}-\alpha_0\boldsymbol{\kappa}(\boldsymbol{\gamma}_0)\|^2\right]}{\sigma_n^4} \\ &= \frac{N}{\sigma_n^2} - \frac{1}{\sigma_n^4}\left(\|\mathbf{r}_0\|^2 + N\sigma_n^2\right)\Big|_{\bar{\sigma}_n^2=\sigma_0^2} = 0, \end{aligned} \tag{14}$$

where $\mathbf{r}_0 \triangleq \mathbf{r}(\boldsymbol{\theta}_0) = \bar{\alpha}\boldsymbol{\mu}(\bar{\boldsymbol{\eta}}) - \alpha_0\boldsymbol{\kappa}(\boldsymbol{\gamma}_0)$, we immediately get

$$\sigma_0^2 = \bar{\sigma}_n^2 + \|\mathbf{r}_0\|^2/N. \tag{15}$$

As shown in [15], for relatively short coherent integration time and realistic acceleration, we have that the pseudo-true parameters are given by:

$$\alpha_0 \approx \bar{\alpha}, \tau_0 = \bar{\tau}, b_0 = \bar{b} + \bar{d}T_e, \tag{16}$$

where T_e is the integration time. The results in eqs. (15) and (16) show that the pesudo-true parameters are different from the true ones except for $\bar{\tau}$ and approximately for $\bar{\alpha}$.

IV. CLOSED FORM EXPRESSION FOR THE MCRB

In this section, the MCRB on the estimation of the pseudo-true parameter vector $\boldsymbol{\phi}_0 \in \Psi$ is presented. This result can be derived from the general formula introduced in [24], where it is shown that the MCRB can be calculated from the product of two matrices $\mathbf{A}(\boldsymbol{\phi}_0)$ and $\mathbf{B}(\boldsymbol{\phi}_0)$ as follows:

$$\mathbf{MCRB}(\boldsymbol{\phi}_0) = \mathbf{A}(\boldsymbol{\phi}_0)^{-1}\mathbf{B}(\boldsymbol{\phi}_0)\mathbf{A}(\boldsymbol{\phi}_0)^{-1}, \tag{17}$$

According to the misspecified signal model introduced in Sec. II-B, the assumed pdf is given by $f_{\mathbf{x}}(\mathbf{x};\boldsymbol{\phi}) = \mathcal{CN}(\alpha\boldsymbol{\kappa}(\boldsymbol{\gamma}), \sigma_n^2\boldsymbol{I}_N)$. This is a particular case of the Scenario 1 in [24, Sec. 3.2] and consequently the matrices $\mathbf{A}$ and $\mathbf{B}$, needed to evaluate the MCRB, can be obtained from eq. [24, Eq. (34)] and [24, Eq. (30)], respectively. By indicating as $\delta[\cdot]$ the Kronecker delta, the following simplifications are in order:

- S1 The matrix, that in the case under study becomes a scalar, in [24, Eq. (19)] can be expressed as $\mathbf{P}_i^0 = \sigma_0^{-2}\delta[i-1]$ for $i \in \{1,2,3,4,5\}$.
- S2 The matrix (scalar in our case) in [24, Eq. (25)] is uniformly equal to 0, i.e. $\mathbf{P}_{ij}^0 = 0\ \forall i,j \in \{1,2,3,4,5\}$.
- S3 The matrix in [24, Eq. (23)] is given by $\mathbf{S}_i^0 = \sigma_0^{-4}\delta[i-1]$ for $i \in \{1,2,3,4,5\}$.
- S4 The term $\mathrm{tr}(\mathbf{S}_i^0\boldsymbol{\Sigma})$ in [24, Eq. (23)] can be obtained as $\mathrm{tr}(\mathbf{S}_1^0\boldsymbol{\Sigma}) = S_1^0\bar{\sigma}_n^2 = (\bar{\sigma}_n^2\sigma_0^{-4})\delta[i-1]$ for $i \in \{1,2,3,4,5\}$.
- S5 The term $\mathrm{tr}(\mathbf{S}_i^0\boldsymbol{\Sigma}\mathbf{S}_j^0\boldsymbol{\Sigma})$ in [24, Eq. (23)] can be evaluated as $\mathrm{tr}(\mathbf{S}_i^0\boldsymbol{\Sigma}\mathbf{S}_j^0\boldsymbol{\Sigma}) = S_i^0 S_j^0\bar{\sigma}_n^4 = (\bar{\sigma}_n^4\sigma_0^{-8})\delta(i-1)\delta(j-1)$, $\forall i,j \in \{1,2,3,4,5\}$.

By making use of S1 - S5, the general expression of the matrix $\mathbf{B}(\boldsymbol{\phi}_0)$ given in [24, Eq. (31)] can be simplified as:

$$\mathbf{B}(\boldsymbol{\phi}_0) = \begin{pmatrix} N\bar{\sigma}_n^4\sigma_0^{-8}(E\{\mathcal{Q}^2\}-1) & \mathbf{0}_{1\times4} \\ \mathbf{0}_{4\times1} & \frac{2\bar{\sigma}_n^2}{\sigma_0^4}\mathbf{C}(\boldsymbol{\theta}_0) \end{pmatrix} \tag{18}$$

where the matrix $\mathbf{C}(\boldsymbol{\phi}_0)$ is given by:

$$[\mathbf{C}(\boldsymbol{\theta}_0)]_{i,j} \triangleq \sum_{k=N_1}^{N_2} \Re\left\{\left[\frac{\partial(\alpha_0\kappa_k(\boldsymbol{\gamma}_0))}{\partial\theta_i}\right]^* \frac{\partial(\alpha_0\kappa_k(\boldsymbol{\gamma}_0))}{\partial\theta_j}\right\} \tag{19}$$

Moreover, by exploiting S1 - S5, it is immediate to verify that the matrix $\mathbf{A}(\boldsymbol{\phi}_0)$ given in [24, Eq. (34)], we get:

$$\mathbf{A}(\boldsymbol{\phi}_0) = \begin{pmatrix} A_{11}(\boldsymbol{\phi}_0) & \mathbf{0}_{1\times4} \\ \mathbf{0}_{4\times1} & \frac{2}{\sigma_0^2}\mathbf{D}(\boldsymbol{\theta}_0) \end{pmatrix} \tag{20}$$

where $A_{11}(\boldsymbol{\phi}_0) = N\sigma_0^{-2}(\sigma_0^{-2} - 2||\mathbf{r}_0||^2 - 2\sigma_0^{-4}\bar{\sigma}_n^2)$ and the matrix $\mathbf{D}(\boldsymbol{\theta}_0)$ is given by:

$$[\mathbf{D}(\boldsymbol{\theta}_0)]_{i,j} = \sum_{k=N_1}^{N_2} \Re\left\{[\mathbf{r}_0]_k^* \frac{\partial^2(\alpha_0\kappa_k(\boldsymbol{\gamma}_0))}{\partial\theta_i\partial\theta_j}\right\} - [\mathbf{C}(\boldsymbol{\theta}_0)]_{i,j} \tag{21}$$

Finally, the MCRB can be expressed as:

$$\begin{aligned} \mathbf{MCRB}(\boldsymbol{\phi}_0) &= \mathbf{A}(\boldsymbol{\phi}_0)^{-1}\mathbf{B}(\boldsymbol{\phi}_0)\mathbf{A}(\boldsymbol{\phi}_0)^{-1} \\ &= \begin{pmatrix} \frac{\sigma_n^4(E\{\mathcal{Q}^2\}-1)}{N\sigma_0^4(\sigma_0^{-2}-2||\mathbf{r}_0||^2-2\sigma_0^{-4}\bar{\sigma}_n^2)^2} & \mathbf{0}_{1\times4} \\ \mathbf{0}_{4\times1} & \mathbf{MCRB}(\boldsymbol{\theta}_0) \end{pmatrix} \end{aligned} \tag{22}$$

$$\mathbf{MCRB}(\boldsymbol{\theta}_0) = \frac{\bar{\sigma}_n^2}{2}\mathbf{D}(\boldsymbol{\theta}_0)^{-1}\mathbf{C}(\boldsymbol{\theta}_0)\mathbf{D}(\boldsymbol{\theta}_0)^{-1} \tag{23}$$

which yields to the classical misspecified MCRB of the Gaussian scenario, proving that asymptotic estimation performance of $\boldsymbol{\theta}_0$ are similar to the pseudotrue parametess considering a true signal model following a Gaussian model [15].

A. Closed-Form MCRB Expression for a Band-Limited Signal

A compact expression of $\mathbf{MCRB}(\boldsymbol{\theta}_0)$, that depends only on the baseband signal samples, was recently derived in [15], $\mathbf{C}(\boldsymbol{\theta}_0)$ in a matrix form can be expressed as

$$\mathbf{C}(\boldsymbol{\theta}_0) = F_s\Re\left\{\mathbf{Q}\mathbf{W}\mathbf{Q}^H\right\}, \tag{24}$$

where

$$\mathbf{Q} = \begin{bmatrix} j\rho_0 & 0 & 0 \\ 1 & 0 & 0 \\ jw_c\rho_0 b_0 & 0 & -\rho_0 \\ 0 & -jw_c\rho_0 & 0 \end{bmatrix},\ \mathbf{W} = \begin{bmatrix} w_1 & w_2 & w_3^* \\ w_2 & W_{2,2} & W_{3,2}^* \\ w_3 & W_{3,2} & W_{3,3} \end{bmatrix}, \tag{25}$$

with $\mathbf{W}$ derived in [16],

$$w_1 = \frac{1}{F_s}\mathbf{a}^H\mathbf{a}, \quad w_2 = \frac{1}{F_s^2}\mathbf{a}^H\mathbf{D}\mathbf{a}, \quad w_3 = \mathbf{a}^H\mathbf{\Lambda}\mathbf{a}, \tag{26}$$

$$W_{3,2} = \frac{1}{F_s}\mathbf{a}^H\mathbf{D}\mathbf{\Lambda}\mathbf{a},\ W_{2,2} = \frac{1}{F_s^3}\mathbf{a}^H\mathbf{D}^2\mathbf{a},\ W_{3,3} = F_s\mathbf{a}^H\mathbf{V}\mathbf{a}.$$

with $\mathbf{a}$, the baseband samples vector, $\mathbf{D}$, $\mathbf{\Lambda}$ and $\mathbf{V}$ defined as,

$$\mathbf{a} = (\ldots,\, a(nT_s),\, \ldots)^\top_{N_1 \le n \le N_2}, \tag{27a}$$

$$\mathbf{D} = \text{diag}\,(\ldots,\, n,\, \ldots)_{N_1 \le n \le N_2}, \tag{27b}$$

$$(\mathbf{\Lambda})_{n,n'} = \left| \begin{array}{l} n' \neq n : \frac{(-1)^{|n-n'|}}{n-n'} \\ n' = n : 0 \end{array} \right. \tag{27c}$$

$$(\mathbf{V})_{n,n'} = \left| \begin{array}{l} n' \neq n : (-1)^{|n-n'|}\ \frac{2}{(n-n')^2} \\ n' = n : \frac{\pi^2}{3} \end{array} \right. \tag{27d}$$

On the other hand, $\mathbf{D}(\boldsymbol{\theta}_0)$ in a matrix form can be expressed as [15]

$$\mathbf{D}(\boldsymbol{\theta}_0) = F_s\rho\Re\{\boldsymbol{\chi}\}, \tag{28}$$

with $\boldsymbol{\chi} = -\mathbf{Q}_e\mathbf{W}_e\mathbf{Q}_e^H$ and

$$\mathbf{Q}_e = \begin{bmatrix} -j\rho_0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & -1 \\ -j\rho_0\omega_c b & 0 & 0 & \rho_0 & 0 \\ 0 & 0 & j\omega_c\rho_0 & 0 & 0 \end{bmatrix}, \tag{29}$$

$$\mathbf{W}_e = \begin{bmatrix} w_{e_1} & 0 & w_{e_2} & w_{e_3} & 0 \\ 0 & w_1 & 0 & 0 & 0 \\ w_{e_2} & 0 & W_{e_{2,2}} & W_{e_{3,2}} & 0 \\ w_{e_3} & 0 & W_{e_{3,2}} & w_{e_M} & 0 \\ 0 & 0 & 0 & 0 & w_{e_1} \end{bmatrix}, \tag{30}$$

$$w_{e_1} = w_1 - j\omega_c\bar{d}T_e w_2, \quad w_{e_2} = w_2 - j\omega_c\bar{d}T_e W_{2,2}, \tag{31}$$
$$w_{e_3} = w_3 - j\omega_c\bar{d}T_e W_{3,2}, \quad W_{e_{3,2}} = W_{3,2} - j\omega_c\bar{d}T_e W_{4,3},$$
$$W_{e_{2,2}} = W_{2,2} - j\omega_c\bar{d}T_e W_{4,2}, \quad w_{e_M} = -W_{3,3} - j\omega_c\bar{d}T_e w_{M,2}$$

with $W_{4,3} = \frac{1}{F_s^2}\left(\mathbf{s}^H\mathbf{D}\mathbf{\Lambda}\mathbf{D}\mathbf{s} - \mathbf{s}^H\mathbf{D}\mathbf{s}\right)$, $W_{4,2} = \frac{1}{F_s^4}\left(\mathbf{s}^H\mathbf{D}^3\mathbf{s}\right)$ and $w_{M,2} = -\mathbf{s}^H\mathbf{D}\mathbf{V}\mathbf{s}$, both derived in [19].

V. Validation and Discussion

We consider a scenarios where a GPS L1 C/A signal [10] is received by a GNSS receiver which assumes that the noise follows a standard centered normal distribution. Then, we set a true signal model where the noise is distributed according to a complex centered Generalized Gaussian (GG) distribution , [14, Sec. 4.6.1.2] with exponent $s > 0$ and scale $b > 0$, where s is a parameter controlling the level of non-Gaussianity. The second-order modular variate $\mathcal{Q}$ of a GG distribution is given by $\mathcal{Q} =_d G^{1/s}$ where G is a Gamma distributed random variable with parameter $1/s$ and b, i.e. $G \sim \text{Gam}(1/s, b)$ [23, Sec. IV.B]. In order to satisfy the constraint $E\{\mathcal{Q}\} = 1$ (see section IV), we set $b = \left(\frac{\sigma_n^2\Gamma(1/s)}{\Gamma(2/s)}\right)^s$ where σ_n^2 depends on the signal to noise ratio at the output of the match filter SNR_{out}. The SNR_{out} is defined as:

$$SNR_{out} = \frac{|\alpha|^2\mathbf{a}^H\mathbf{a}}{\sigma_n^2}. \tag{32}$$

We set the acceleration $d = 50g$, with $g = 9.81m/s^2$, and the integration time to $T_e = 5ms$, i.e. 5 GPS L1 C/A sequences. Moreover, complex centered Generalized Gaussian distributions with $s = \{0.2, 2\}$ has been used as a true model. The MMLE for the joint estimation of the time-delay and Doppler can be expressed as [1]:

$$\hat{\boldsymbol{\gamma}} = \arg\max_{\boldsymbol{\gamma}} \left\|\mathbf{\Pi}_{\boldsymbol{\kappa}(\boldsymbol{\gamma})}\boldsymbol{x}\right\|^2 \tag{33}$$

For this algorithm we can verify that the delay estimation is unbiased. On the other hand, the Doppler estimate is biased with $\Delta b = dT_e$. The root mean square error (RMSE) results of the MMLE for the parameters of interest $\boldsymbol{\eta}^T = [\tau, b]$ are shown in Figs. 1 and 2 w.r.t. the SNR_{out} and considering the following setup: a GNSS receiver with sampling frequency $F_s = 2$ MHz and the number of Monte Carlo is set to 1000 iterations. The results clearly demonstrate that the RMSE, represented as $\sqrt{MSE}$, for the pseudotrue parameter converges towards the asymptotic estimation performance derived in Section IV. These findings validate the theoretical derivation. It's worth noting that the square root of the misspecified Cramér-Rao bound ($\sqrt{MCRB_\tau}$) for the time-delay is equivalent to the square root of the Cramér-Rao bound ($\sqrt{CRB_\tau}$) of the time-delay. This is due to the fact that the time-delay bias is this particular case is zero. Moreover, we remind that the CRB is equal whether acceleration is considered or not within the signal model [19]. Thus, time-delay MMLE estimates reduces the complexity at the receiver and provides the same asymptotic performance than the MLE. On the other hand, we can verify that the $\sqrt{MCRB_b}$ for the Doppler differs from the Doppler $\sqrt{CRB_b}$. Note that the $\sqrt{CRB_b}$ represents the asymptotic performance of the joint Doppler and acceleration estimation. By comparing these two bound along with the bias Δ_b, we can verify that the MMLE, which is less computationally consuming, can improve the estimation performance of the MLE for certain SNR_{out} regimes. Moreover, we can observe that the MCRB is the same as in the case where the true error distribution is a complex Gaussian [15]. It is important to emphasize that the earlier theoretical findings apply to all CES-distributed true noise models, not just the GG distribution. A formal explanation of this observation is grounded in semiparametric theory (as outlined in [25, Sec. IV;B] and [26, Sec. III.B]). A more detailed and comprehensive explanation will be offered in future research endeavors.

VI. Conclusion

The objective of this paper is to present novel insights to the theory of time-delay and Doppler estimation. Specifically, we derive asymptotic performance expressions MCRB for a non-standard scenario where a receiver decides to carry out the estimation of a subset of parameters of interest in order to reduce the computational complexity. Moreover, the true noise model conforms to a centered CES distribution, while the receiver assumes that the noise model adheres to a

[1]Let $S = span\,(\mathbf{A})$, with $\mathbf{A}$ a matrix, be the linear span of the set of its column vectors. The orthogonal projector over S is $\mathbf{\Pi}_{\mathbf{A}} = \mathbf{A}\left(\mathbf{A}^H\mathbf{A}\right)\mathbf{A}^H$.

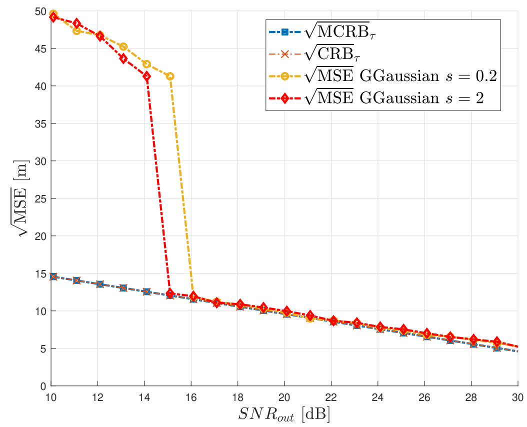

Fig. 1. RMSE of the MMLE of the time-delay considering complex centered GG dist. with $s = \{0.2, 2\}$. The sampling frequency is set to $F_s = 2$ MHz and the integration time is set to $T = 5$ms.

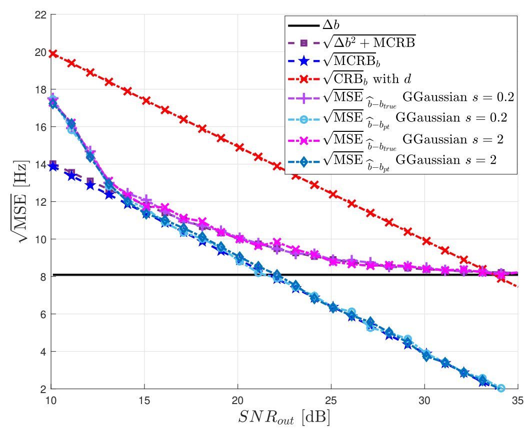

Fig. 2. RMSE of the MMLE of the Doppler considering complex centered GG dist. with $s = \{0.2, 2\}$. The sampling frequency is set to $F_s = 2$ MHz and the integration time is set to $T = 5$ms.

centered complex normal distribution. The main conclusion is that the asymptotic estimation performance of the parameters of interests is independent of the true noise distribution and it is the fact of considering an alternative signal model that can generate perturbation in the estimation performance. For this scenario, we have observed that the time-delay MMLE is unbiased and converges to the CRB, i.e. is indepedent of the acceleration value. On the other hand, the Doppler estimate is biased, and converges to the MCRB. Moreover, we have shown that the MCRB is indepedent of the true noise distribution.

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