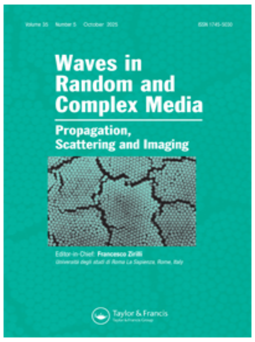


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# Causal, stable and homogeneous formulas for acoustic and ultrasonic propagation through atmosphere

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# Causal, stable and homogeneous formulas for acoustic and ultrasonic propagation through atmosphere

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**ABSTRACT**
This paper addresses the propagation of acoustics or ultrasonics waves through atmosphere and the causality property. The physicists community seems to agree with the following sentence:
*… empirical observation indicates that such systems are indeed causal even though the transfer function may not be a causal transform.*
We explain that the complex gain is not causal when not properly chosen, and that this issue can be addressed.

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Acoustics; ultrasonics; atmospheric propagation; attenuation; dispersion; causality

## 1. Preamble

This paper addresses the propagation of acoustics and ultrasonics through the atmosphere. The widely accepted model is a linear invariant filter (LIF) $\mathcal{H}$ with a 'complex gain' $H(f)$ in the form

$$H(f) = \exp\left[\left(-\alpha f^2 + imf\right) - \frac{a_2 f^2}{b_2 + f^2} - \frac{a_3 f^2}{b_3 + f^2}\right] \tag{1}$$

where $\alpha, a_j, b_j > 0$, and where $m > 0$ [1–3]. $H(f)e^{-2i\pi ft}$ is the response of the medium to $e^{-2i\pi ft}$ ($f \in \mathbb{R}$ is the frequency). $m/2\pi$ is the propagation time when variations with $f$ are ignored. $H(f)$ is the product of three exponential functions. Then the LIF $\mathcal{H}$ can be viewed as a series of three LIF $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$, in any order.

Actually, formula (1) does not take into account a small 'dispersion'. It is a variation of the wave celerity with the frequency, which is explainable and measurable provided stable and accurate conditions [4–7]. Measurements justify a complementary term in (1) of the form

$$\frac{ic_2 f}{b_2 + f^2} + \frac{ic_3 f}{b_3 + f^2}. \tag{2}$$

Indices 2 and 3 correspond to contributions of $N_2$ and $O_2$, main components of the atmosphere. The 'Gaussian' term in $\exp[-\alpha f^2 + imf]$ in (1) is common to propagation in many gases and liquids [6], and was generalized in the 'frequency power law' model [8,9].

## 2. Causality

Linear invariant filters (LIF) summarize linear transformations $\mathcal{H}$ as (for instance)

$$\begin{gathered} e^{-2i\pi ft} \rightarrow_{\mathcal{H}} H(f)\, e^{-2i\pi ft} \\ H(f) = |H(f)|\, e^{2i\pi f\tau_f} \end{gathered} \tag{3}$$

when $f$ (the frequency) belongs to some set $\Delta$, and $t \in \mathbb{R}$. This means that a monochromatic wave is changed in a monochromatic wave of same frequency $f$, with amplitude and phase following $H(f)$ (the 'complex gain').

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$\tau_f$ is chosen from physical considerations in the set of values $\tau_f + k/f, k \in \mathbb{Z}$. Studied phenomena are real. This means that $\mathcal{H}$ realizations are the real or the imaginary part of (3) or real linear combinations of both.

It is a common practice to consider a LIF as a 'blackbox' with input $in(t)$ and output $out(t)$ linked by some formulas or some algorithms (for instance, a convolution product).

We assume the existence of the finite measure $\mathfrak{h}$ such that

$$H(f) = |H(f)| \, e^{2i\pi f \tau_f} = \int_{\mathbb{R}} e^{2i\pi ft} \, d\mathfrak{h}(t) \tag{4}$$

where $|H(f)|$ is the 'attenuation' of the pure wave $e^{-2i\pi ft}$ by the medium and $\tau_f$ its 'propagation time'. If we can write in some sense $\mathfrak{h}$ under the shape

$$d\mathfrak{h}(t) = h(t) \, dt \tag{5}$$

$h(t)$ is the 'impulse response'. It may be the case where $h(t)$ is not an ordinary function (for instance it is a usual function added to pulses). When $h(t)$ is a 'regular function', some Fourier theory can be applied (in the $\mathbf{L}_1$ or $\mathbf{L}_2$ case, simple inversion formulas are available).

Causality property translates 'effect does not precede cause' into:

$$\text{if } in(t) = 0 \quad \text{for } t < t_0, \qquad \text{then } out(t) = 0 \quad \text{for} t < t_0 \tag{6}$$

whatever $t_0$. For example the property is verified when (4) can be written as (wathever $\varepsilon > 0$)

$$H(f) = \int_{-\varepsilon}^{\infty} e^{2i\pi ft} \, d\mathfrak{h}(t). \tag{7}$$

Generally, causality is linked to Kramers–Krönig formulas or to Paley–Wiener conditions, in the $\mathbf{L}_2$ framework [10–12]. Then, both Fourier transforms $h(t)$ and $H(f)$ are $\mathbf{L}_2$ functions. Formula (1) is not causal, and remains non-causal when completed at best [7], though $H(f)$ remains in $\mathbf{L}_2$. The lack of causality comes mainly from the Gaussian term. Nevertheless, these formulas are widely used without real hindrance in physical applications, because the part of the impulse response in $\mathbb{R}_-$ is always weak.

The question which is addressed in this paper is: *how to modify* (1) *in such a way*$H(f)$*becomes causal*?

Section 3 defines a class of functions which explains the addition of (2) to (1) and which is based on robust physical and mathematical arguments. Section 4 reaches a solution in the same class which approximates a Gaussian by a causal function, and which gives stable, coherent and homogenized formulas. These results are applied to the propagation of acoustics and ultrasonics through the atmosphere.

## 3. A family of causal filters

### 3.1. The property

Let the complex gain $H_b(f)$ be defined as ($a, b > 0, 0 \le \varepsilon \le 1$, real $m$)

$$H_b(f) = \exp\left[-a\frac{f^2 + if(2\varepsilon - 1)\sqrt{b}}{b + f^2} + imf\right]. \tag{8}$$

Then $H_b(f)$ is causal if and only if $\varepsilon = 0, m \ge 0$.

A proof is given in the appendix, from considerations about properties of characteristic functions (c.f.) of the probability calculus.

### 3.2. The impulse response

For $\varepsilon = 0, m = 0$, the impulse response $h_b(t)$ of (8) cancels out when $t < 0$, and verifies $h_b(t) > 0$ for $t > 0$ (see appendix). We have the relations

$$H_b(f) = H_1\left(f/\sqrt{b}\right)$$
$$h_b(t) = \sqrt{b} h_1\left(t\sqrt{b}\right)$$

which allows to only consider the case $b = 1$.

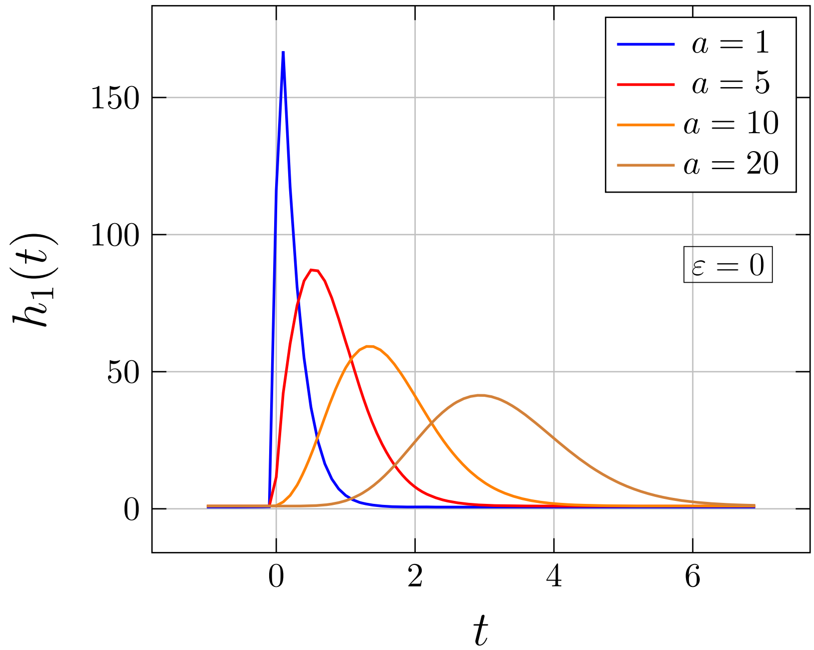


**Figure 1.** Continuous part of the causal impulse response $h_1(t)$ corresponding to $\epsilon = 0$ (formulas (6) and (7)).

Actually, $h_1(t)$ is not an ordinary function because $\lim_{f\to\infty}|H_1(f)| > 0$. $h_1(t)$ is the sum of a regular function and a 'mass' at the origin point. Figure 1 depicts (causal) impulse responses $h_1(t)$ for several values of $a$, by computing the following integrals ($\varepsilon = 0$)

$$h_1(t) - e^{-a}\delta(t) = 2e^{-a}\int_0^{\infty}\left[\theta_1(f) - \cos 2\pi ft\right] df.$$
$$\theta_1(f) = e^{a/(1+f^2)} \cos\left[2\pi f\left(t - \frac{a}{2\pi(1+f^2)}\right)\right] \tag{9}$$

We verify the causal property, i.e. $h_b(t) = 0, t < 0$. Furthermore, we also have $h_b(t) > 0, t > 0$, whatever $b$.

By comparison, Figure 2 shows

$$h_1(t) - e^{-a}\delta(t) = 2e^{-a}\int_0^{\infty}\left[e^{a/(1+f^2)} - 1\right]\cos 2\pi ft\, df \tag{10}$$

which is the impulse response linked to the real gain corresponding to $\varepsilon = 0.3, m = 0$ in (8). Results are no longer causal, and it is always the case when $\varepsilon \neq 0$. In both cases, a mass equal to $e^{-a}$ has to be added at the origin point.

## 4. Causal approximation of a Gaussian filter

### 4.1. The model

We define the Gaussian filter by the complex gain (it is a c.f.)

$$H(f) = \exp\left[-\alpha f^2 + imf\right], \quad \alpha > 0, m \geq 0. \tag{11}$$

It is wellknown that the corresponding probability density is

$$\underline{h}(t) = \frac{1}{2\sqrt{\alpha\pi}} \exp\left[\frac{-(t-m)^2}{4\alpha}\right] \tag{12}$$

and the impulse function (because of the slight difference in the definition of Fourier transforms, see appendix)

$$h(t) = \sqrt{\frac{\pi}{\alpha}} \exp\left[\frac{-\pi^2}{\alpha}\left(t - \frac{m}{2\pi}\right)^2\right]. \tag{13}$$

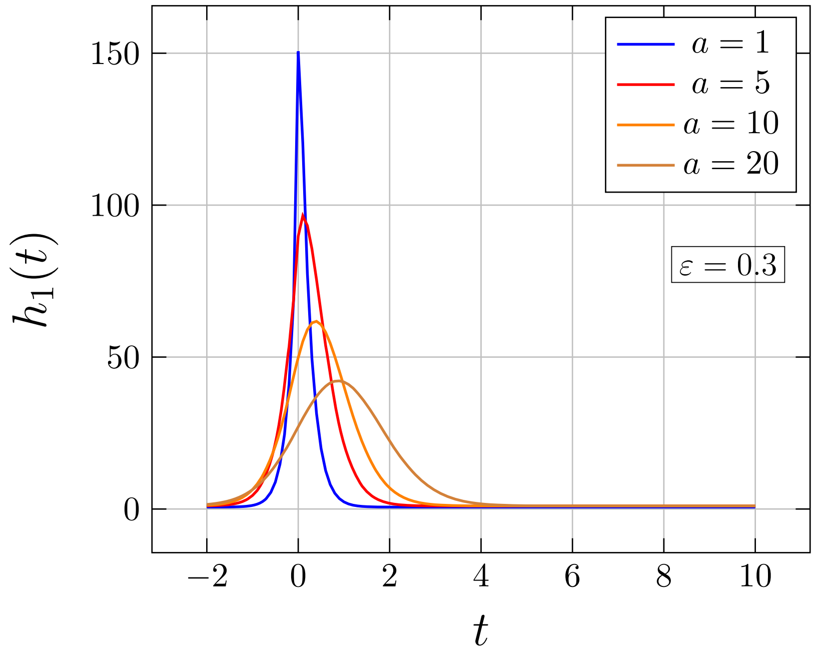


**Figure 2.** Continuous part of the non-causal impulse response $\widetilde{h}_1(t)$ corresponding to $\epsilon = 0.3$ (formula (7)).

$h(t)$ and $\underline{h}(t)$ are linked by the relation

$$h(t) = 2\pi\,\underline{h}(2\pi t)$$

and they are Gaussian centered at the point $m/2\pi$ and $m$. They do not cancel out, and then this filter is never causal. Actually, an experimental plan works inside a limited frequential interval $\Delta = (0, f_1)$. So, we have to find a filter with a complex gain arbitrarily close to (11) in $\Delta$, and with a continuation outside $\Delta$ such that the result fulfills the conditions of causality.

We propose to replace $H(f)$ defined in (11) by

$$\widetilde{H}(f) = \exp\left[\frac{-\alpha b f^2}{b + f^2} + \mathrm{i}f\frac{\alpha b^{3/2}}{b + f^2}\right] \tag{14}$$

with $b$ such that

$$\frac{f_1^2}{b} \ll 1. \tag{15}$$

$\widetilde{H}(f)$ is a causal characteristic function (see Appendix). If condition (15) is fulfilled, then $|\widetilde{H}(f)|$ and $|H(f)|$ are interchangeable on $\Delta$. Moreover ($\widetilde{t}_f$ is the corresponding propagation time):

$$2\pi\widetilde{t}_f = \frac{\alpha\sqrt{b}}{1 + (f^2/b)}. \tag{16}$$

From (15), $2\pi\widetilde{t}_f$ remains close to $a\sqrt{b}$ in $\Delta$ and $2\pi t_f = m$. Taking (for example) $b$ such as

$$m = \alpha\sqrt{b} \tag{17}$$

makes $\widetilde{H}(f)$ and $H(f)$ interchangeable on $\Delta$, provided that the obtained value of $b = (m/\alpha)^2$, knowing $(m, \alpha)$, verifies (15).

### 4.2. Moments

$H(f)$ and $\widetilde{H}(f)$ are the c.f. of the probabilities $\underline{h}(t)$ and $\underline{\widetilde{h}}(t)$. The derivatives of $H(f)$ and $\widetilde{H}(f)$ defined in (11) and (14) exist and they are linked to the moments $m_1, m_2, \widetilde{m}_1, \widetilde{m}_2 \ldots$ of $\underline{h}(t)$ and $\underline{\widetilde{h}}(t)$ [13,14]:

$$H'(0) = \mathrm{i}m_1 \quad H''(0) = -m_2$$
$$\widetilde{H}'(0) = \mathrm{i}\widetilde{m}_1 \quad \widetilde{H}''(0) = -\widetilde{m}_2$$

In comparison with $m, \alpha, b$, we have, from (11) for the first line and from (14) for the second one,

$$H'(0) = \mathrm{i}m \quad H''(0) = -m^2 - 2\alpha$$
$$\widetilde{H}'(0) = \mathrm{i}\alpha\sqrt{b} \quad \widetilde{H}''(0) = -\alpha^2 b - 2\alpha$$

Then, defining $b$ by (17) leads to the equality of the mean and the variance of probability laws. The property fails for the third moment, because $\widetilde{H}(f)$ is not symmetric:

$$\begin{cases} m_3 = m^3 + 6m\alpha \\ \widetilde{m}_3 = m^3 + 6m\alpha + \left(6\alpha^2/m\right). \end{cases}$$

The last term $6\alpha^2/m$ characterizes the 'skewness' of $\widetilde{H}(f)$. When $\underline{h}(t)$ is replaced by $\underline{\widetilde{h}}(t)$ (or $h(t)$ by $\widetilde{h}(t)$), the part in $\mathbb{R}_-$ is suppressed (it is the causality), and the part in $\mathbb{R}_+$ is modified so that the two first moments (the mean and the variance) do not change.

### 4.3. Distances

In dB by unit length (100 m in [1]), absorptions (inverses of attenuations) are defined as ($20\log_{10} e = 8.686$)

$$\begin{cases} d_H(f) = 8.686\alpha f^2 \\ d_{\widetilde{H}}(f) = 8.686\alpha b f^2 \left[b + f^2\right]^{-1}. \end{cases}$$

In logarithmic coordinates, $d_H(f)$ is a line of slope 2. For $b$ large enough, $d_{\widetilde{H}}$ is composed by a part confused with $d_H(f)$ for $f \in \Delta$, followed by an elbow and ended by a horizontal asymptote. Figure 3 illustrates the property for $\alpha = 1, f_1 = 10^6$, and different values of $b$. The ordinate of the asymptote is equal to $\log[8.686\alpha b]$.

In this representation, we can define the distance $d_b$ between the curves by

$$d_b = \sup_{f\in\Delta} \log \frac{d_H(f)}{d_{\widetilde{H}}(f)} = \log\left(1 + \frac{f_1^2}{b}\right)$$

Provided that $b$ is taken large enough, both curves will be confused on $\Delta$.

Tables take into account the absorption and the celerity. About the latter, we have a constant celerity $v_f = 1/t_f$ in the Gaussian case and a weakly increasing one for $\widetilde{H}(f)$. From (16) and (17)

$$\frac{\widetilde{v}_{f_1} - \widetilde{v}_0}{\widetilde{v}_0} = \frac{f_1^2}{b}, \widetilde{v}_0 = \frac{2\pi}{m}.$$

where $\widetilde{v}_f$ is the celerity corresponding to $\widetilde{H}(f)$. Consequently, at given $(m, \alpha)$, the identification of both complex gains is possible but in a limited interval of frequencies.

### 4.4. Gaps with causality

Because $\widetilde{h}(0) = 0$, a gap with the causality may be defined by $h(0)$, or by the following probability linked to the law $N(0, 1)$ :

$$\begin{aligned} h(0) &= \sqrt{\frac{\pi}{\alpha}} \exp\left[-\frac{m^2}{4\alpha}\right] \\ \Phi\left(-m/\sqrt{2\alpha}\right) &= \int_{-\infty}^{-m/\sqrt{2\alpha}} \frac{1}{\sqrt{2\pi}} e^{-u^2/2}\, du = \int_{-\infty}^{0} h(t)\, dt. \end{aligned} \tag{18}$$

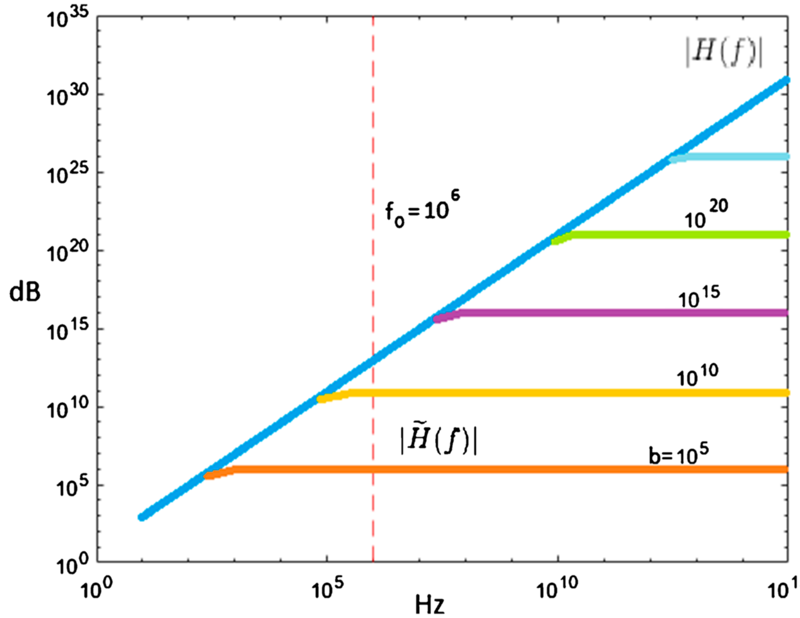


**Figure 3.** Attenuation $|H(f)|$ of the Gaussian filter (formula (11)) and its causal approximation $|\widetilde{H}(f)|$ (formula (14)) for $\alpha = 1$ and different values of $b$.

## 5. Application to the atmospheric propagation

Atmospheric propagation through atmosphere is a framework which was extensively studied. We find the Gaussian attenuation in a wellknown paper of Stokes as particular case 15, 1845]). More developed formulas such as (1), or including the dispersion, appear in the 1930's. Official recommendations [3], formulas and tables appear later [5,7]. All exhibit a Gaussian component, in contradiction with the causality.

### *5.1. A causal formula*

Formula (1) is currently accepted for the attenuation. It hides the dispersion, which is explained for example in [6,7], and tabulated in [5]. Finally, the 'official' formula has the form

$$H(f) = \exp\left[\left(-\alpha f^2 + imf\right) + a_2\frac{-f^2 + ifc_2}{b_2 + f^2} + a_3\frac{-f^2 + ifc_3}{b_3 + f^2}\right]. \tag{19}$$

We see three components with one which is not causal (the Gaussian in parentheses), and the others which are causal only if

$$c_2 = \sqrt{b_2}, c_3 = \sqrt{b_3}.$$

$H(f)$ is not causal. This drawback can be avoided using the formula (see Section 4)

$$\begin{gathered}\widetilde{H}(f) = \exp\left[\sum_{k=1}^{3} g_k(f)\right] \\ g_k(f) = a_k\frac{-f^2 + if\sqrt{b_k}}{b_k + f^2}, \quad k = 1,2,3 \\ b_1 = (m/\alpha)^2, a_1 = \alpha b_1\end{gathered} \tag{20}$$

provided that (15) is verified. Each $g_k(f)$ leads to the causal filter of spectral gain $\exp g_k(f)$ (see Sections 3 and 4). Then, $\widetilde{H}(f)$ defines a causal filter:

$$\widetilde{h}(t) = 0, \quad t < 0.$$

Parameters depend on macroscopic quantities (temperature, pressure, humidity ...). We can see $\widetilde{H}(f)$ as a combination of three successive causal LIF taken in an indifferent order. Real parts define attenuations and imaginary parts define dispersions. Taking into account parameters $(m, \alpha)$, $g_1(f)$ defines the causal filter which replaces the (non-causal) Gaussian filter. The $g_k(f)$ belong to the only causal set in the shape (8). They replace terms in [7], which come from physical considerations.

The propagation time $\widetilde{t}_f$ and the celerity $(\widetilde{v}_f - \widetilde{v}_0)$ are calculated from the sum of the $g_k(t)$ with $m > 0$ :

$$2\pi \widetilde{t}_f = \frac{\alpha b_1^{3/2}}{b_1 + f^2} + \frac{\alpha_2 \sqrt{b_2}}{b_2 + f^2} + \frac{\alpha_3 \sqrt{b_3}}{b_3 + f^2}$$
$$\widetilde{v}_f = 2\pi \left( \frac{\alpha b_1^{3/2}}{b_1 + f^2} + \frac{\alpha_2 \sqrt{b_2}}{b_2 + f^2} + \frac{\alpha_3 \sqrt{b_3}}{b_3 + f^2} \right)^{-1}. \quad (21)$$

### 5.2. Stability

The stability is a very important property that a LIF has to verify. This means that a bounded input leads to a bounded output. Since $\widetilde{H}(f)$ is the concatenation of three filters of the same nature, it is sufficient to verify the property for one of them, $H(f)$. In this case, we have (Equation (28), Section 8.3):

$$H(f) = e^{-\lambda} + \int_0^{\infty} e^{ift} \mu(t)\, dt$$

where $\mu(t)$ is real, continuous, bounded and positive. $H(f)$ is the c.f. of a probability law with a mass $e^{-\lambda}$ at the origin added to a regular positive density $\mu$ on $R_+$. The first part corresponds to the filter $\delta \to e^{-\lambda}\delta$, and the second part is stable because $\mu \in L_1$ (it is the NSC usually invoked). Then, $H(f)$ is stable.

### 5.3. Example

(1) Let us consider the atmospheric propagation up to 1 MHz at 20 °C, 1 atm and 30% of humidity. Tables in [5] deliver series of 51 values of the absorption $-H_{dB}(f)$ and the velocity $v_f$ from 12 Hz to 1 MHz (unities are the second and 100 m). We have, in dB:

$$-H_{dB}(f) = 8.686 \left[ \alpha f^2 + \frac{a_2 f^2}{b_2 + f^2} + \frac{a_3 f^2}{b_3 + f^2} \right] \quad (22)$$

The curve **D** in Figure 4 depicts the absorption, from data in [5]. Parameters of $-H_{dB}(f)$ are obtained by the mean square method and are given in Table 1. Curves $\mathbf{D}_1, \mathbf{D}_2, \mathbf{D}_3$ depict the three components of (21), the Gaussian in $f^2$ and the others which reflect the roles of $N_2$ (index 2) and $O_2$ (index 3).

(2) Whatever the more or less developed version of $H(f)$, $m$ is the mean of the Gaussian. From (16) and (17). we have together

$$2\pi / \widetilde{v}_0 = \alpha \sqrt{b_1} = m$$

which, using data of [5]:

$$\widetilde{v}_0 = 3.438, m = 1.8275$$

and, using Table 1

$$b_1 = \left( \frac{m}{\alpha} \right)^2 \approx 10^{18}$$

which verifies the condition (15):

$$\frac{f_1^2}{b_1} \approx 10^{-6} \ll 1.$$

(3) Figure 5 depicts the evolution of velocity. The curve $\mathbf{E}_1$ depicts $v_f - v_{12}$ from data of [5] and $\mathbf{E}_2$ for its computation through formula (21) and estimations of Table 1. The values of $v_{12}$ and $v_0$ can be confused but the value of $\alpha$ is very sensible. Its value in Table 1 is not accurate enough. The curve $\mathbf{E}_2$ is drawn for $\alpha = 1827 \cdot 10^{-8}$. With this value, both curves are confused up to $10^5$ Hz, and $v_{12} - \widetilde{v}_{12} = 2.8\,\text{cm·s}^{-1}$ to compare with $v_{12} = 343.80\,\text{m·s}^{-1}$ (value given in [5]). The maximum gap is equal to $0.5\,\text{cm·s}^{-1}$ at $10^6$ Hz.

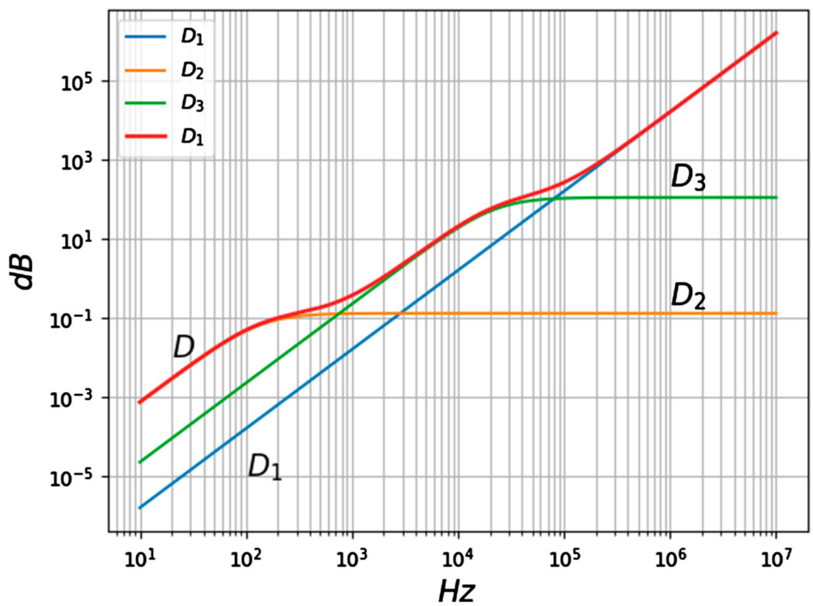


**Figure 4.** $D, D_1, D_2, D_3$ describe the absorption and its components in 'log log' coordinates, under the conditions of Section 5.3.

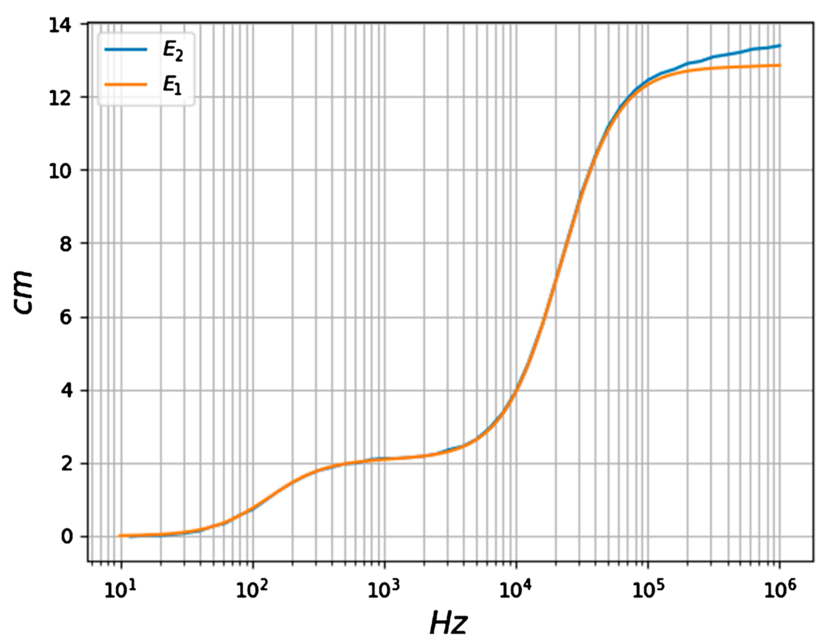


**Figure 5.** $E_1$ describes the increase of celerity $v_f - v_{12}$ [5], $E_2$ describes the increase of celerity $\widetilde{v_f} - \widetilde{v_{12}}$.

**Table 1.** Propagation parameters estimation.

| | | |
|---|---|---|
| $\alpha = 18 \cdot 10^{-10}$ | $a_2 = 0.023$<br>$b_2 = 38 \cdot 10^3$ | $a_3 = 11.5$<br>$b_3 = 36 \cdot 10^7$. |

## 6. Conclusion

Causality is an axiom of physics. It addresses a system input–output which treats quantities which evolve with time (see Section 2). We consider the case where the system is linear and transforms a pure wave into a pure wave of same frequency (it is a 'Linear Invariant Filter'). In this situation, the system is defined by a function of the frequency, the 'complex gain', which rules the changes of amplitude and phase of pure waves. Its Fourier transform is the 'impulse response' which contains the information about the causality.

Atmospheric propagation of acoustics and ultrasonics is summarized in formulas, tables and algorithms which are well accepted. In a first approximation, it is sufficient to consider only an attenuation in the form (1), also observed in propagation through sea water [16,17]. This leads to 3 terms, each of them defining a non-causal filter.

The last two terms of (1) do not belong to the class of 'frequency power laws' [8], widely used in ultrasonic propagation [9,12,18–21]. Adding complex terms of formula (2) in formula (1) can improve the model, taking into account a variation of the transit times with the frequency (a dispersion). In the case studied in Section 5.3, the gap is in the order of 15 cm compared with 340 m (about one second of the trajectory on the frequency interval going to $10^6$ Hz [5,7]). The causality is not far away, and can be reached by a causal approximation of the Gaussian.

In real applications, some function is chosen as model when some cloud of data is well fitted by this function and when the latter has good properties and can easily be handled. Nevertheless, the cloud defines a finite part of some space $\Delta$ (here, a frequency interval of length 1 MHz), and data do not give information in $\overline{\Delta}$ (the complementary of $\Delta$) where conditions of causality appear when applying Kramers–Krönig or Paley–Wiener conditions. Simultaneously, it may be possible to fit the cloud in $\Delta$ by other classes of functions, which have a good behavior when it is necessary. In the present case, we have to find a function which is close to the Gaussian in the finite interval $\Delta = (0, 10^6)$ in Hertz, and which shows a behavior in $\overline{\Delta}$ which is in accordance with causal conditions. Surprisingly and happily, a function which is eligible to replace the Gaussian, has the same shape as the other components of (1), provided that parameters are well-chosen.

More generally, we find the following sentence in the basic reference [12]:

*… empirical observation indicates that such systems are indeed causal even though the transfer function may not be a causal transform.*

We explain that observation can lead to families of transfer functions (and not only one) and we have to choose among them the correct one, which is causal.

We have shown that formula (1) or a more elaborated one (as given in [7]), which models the acoustic and ultrasonic propagation through atmosphere and which is non-causal, can be replaced by a causal one detailed in formulas (20).

Finally, the answer to the question: *does it exist a causal model for the atmospheric propagation?* is: *yes, it exists a causal model which is suitable for each component of the propagation.*

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## Disclosure statement

No potential conflict of interest was reported by the author(s).

## Author contributions

CRediT: **Bernard Lacaze:** Writing – original draft

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## Appendix

### A.1 Linear invariant filter

We start from

$$e^{-2i\pi ft} \to_{\mathcal{G}} G(f)\,e^{-2i\pi ft} \tag{A1}$$

and we use the linearity of the transformation. Schematically, this leads to the sequence

$$\sum_k Q(f_k)\,e^{-2i\pi f_k t} \to_{\mathcal{G}} \sum_k G(f_k)\,Q(f_k)\,e^{-2i\pi f_k t}$$

$$q(t) = \int_{-\infty}^{\infty} Q(f)\,e^{-2i\pi ft}\,df \to_{\mathcal{G}} \int_{-\infty}^{\infty} G(f)\,Q(f)\,e^{-2i\pi ft}\,df$$

provided that $G, Q$ have the good properties. The Fourier transform theory leads to the fundamental relation of LIF by the 'convolution theorem'

$$\mathcal{G}[q](t) = \int_{-\infty}^{\infty} q(u)\,g(t-u)\,du$$

$$g(t) = \int_{-\infty}^{\infty} G(f)\,e^{-2i\pi ft}\,df.$$

$g(t)$ is the 'impulse response' of the LIF $\mathcal{G}$. $g(t)$ has to be real because, in the real world, inputs and outputs are real.

### A.2 Characteristic functions

Let $\mathbf{X}$ be a random variable with $\underline{H}(x) = \Pr[\mathbf{X} < \mathbf{x}]$. The characteristic function $\phi(f)$ of $\mathbf{X}$ is defined as

$$\phi(f) = \mathrm{E}\left[e^{if\mathbf{X}}\right] = \int_{-\infty}^{\infty} e^{ift}\,d\underline{H}(t).$$

In the case of a 'regular' r.v., $(d\underline{H}(t) = \underline{h}(t)\,dt)$ it is a Fourier transform, except for the constant $2\pi$ in the exponent. The study of c.f. covers a whole field of the probability calculus [13]. Particularly, they model propagations following a frequency power law [8,20,21].

### A.3 A class of c.f.

(1) Let us consider the set of independent random variables (r.v) $N, B_n, n > 0$, with the following properties
  (a) $N$ is Poisson of parameter $\lambda > 0$ [14]. This means that, for any integer $n \geq 0$

$$\Pr[N = n] = \frac{\lambda^n}{n!}e^{-\lambda}.$$

(b) r.v. $B_n, n > 0$, follow a law with the probability density

$$g(t) = \begin{cases} (1-\varepsilon)\sqrt{b}\mathrm{e}^{-\sqrt{b}t}, & t > 0 \\ \varepsilon\sqrt{b}\mathrm{e}^{\sqrt{b}t}, & t < 0 \end{cases} \tag{A2}$$

with $b > 0, 0 \le \varepsilon \le 1$. The calculus of the c.f. $\psi_B(f) = \mathrm{E}[\mathrm{e}^{\mathrm{i}fB_n}]$ is straightforward:

$$\psi_B(f) = \frac{b + \mathrm{i}f\sqrt{b}(1-2\varepsilon)}{b + f^2} \tag{A3}$$

(2) The r.v. $C$ is defined as

$$C = \left\{ \begin{matrix} \sum_{k=1}^{N} B_k & \text{when } N \neq 0 \\ 0 & \text{when } N = 0. \end{matrix} \right\}$$

Using conditional probabilities and the independence of r.v. leads to

$$\mathrm{E}\left[\mathrm{e}^{\mathrm{i}fC} | N\right] = \begin{cases} 1 & \text{when } N = 0 \\ \left[\psi_B(f)\right]^N & \text{when } N \neq 0. \end{cases}$$

$$\mathrm{E}\left[\mathrm{e}^{\mathrm{i}fC}\right] = \mathrm{e}^{-\lambda} + \sum_{n=1}^{\infty} \left[\psi_B(f)\right]^n \frac{\lambda^n}{n!} \mathrm{e}^{-\lambda}$$

and then

$$\mathrm{E}\left[\mathrm{e}^{\mathrm{i}fC}\right] = \exp\left[-\lambda + \lambda\psi_B(f)\right]. \tag{A4}$$

(3) From (A1), (A2), and whatever $m \in \mathbb{R}$:

$$\mathrm{E}\left[\mathrm{e}^{\mathrm{i}f(C+m)}\right] = \exp\left[-\lambda\frac{f^2 + \mathrm{i}f\sqrt{b}(2\varepsilon - 1)}{b + f^2} + \mathrm{i}mf\right] \tag{A5}$$

which proves that the $\psi(f)$ defined in (8) are c.f. It is a particular case of the 'De Finetti theorem' 13, th. 5.4.1]).

Now, for $\varepsilon = 0$ (vs $\varepsilon = 1$), the $B_n$ are non-negative (vs non-positive) and then $C$ is one-sided, with positive values for $\varepsilon = 0$ and negative for $\varepsilon = 1$. When $0 < \varepsilon < 1$, we have, from the definition of $C$:

$$\begin{cases} \Pr[C > 0] \ge \lambda\mathrm{e}^{-\lambda}\Pr[B_1 > 0] > 0 \\ \Pr[C < 0] \ge \lambda\mathrm{e}^{-\lambda}\Pr[B_1 < 0] > 0 \end{cases}$$

and then $C$ is not one-sided.

(4) Let assume that $\varepsilon = 0$ and then $C \ge 0$. We have, for any real $k < k' < 0$

$$\Pr\left[k < C < k'\right] \ge \lambda\mathrm{e}^{-\lambda}\Pr\left[k < B_1 < k'\right] > 0.$$

So, the probability law of $C$ does not cancel on $\mathbb{R}^-$(or on $\mathbb{R}^+$ when $\varepsilon = 1$).

To summarize, when $\varepsilon = 0, m \ge 0$, formula (7) defines a c.f. of a non-negative r.v, in the form

$$H(f) = \mathrm{e}^{-\lambda} + \int_0^{\infty} \mathrm{e}^{\mathrm{i}ft}\mu(t)\,\mathrm{d}t \tag{A6}$$

where $\mu(t)$ is a bounded, continuous and positive function. Then, viewed as a complex gain, $H(f)$ is causal.