# TIME-DELAY MAXIMUM-LIKELIHOOD ESTIMATOR UNDER PHASE UNCERTAINTY

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## ABSTRACT

Accurate signal time-delay estimation is critical in localization, sensor fusion, and communication systems. In multi-sensor contexts, where distributed nodes combine measurements, modeling estimation performance under realistic assumptions is key. A common challenge is global phase misalignment, stemming from hardware imperfections at both transmitter and receiver. While some works assume perfect calibration and others treat phase as completely unknown, we propose a middle-ground model where the phase is partially known, i.e., estimated with uncertainty. This approach is particularly relevant in practical multi-sensor scenarios, where each node may experience different phase conditions. The goal is to quantify how an additional measurement of the unknown phase can enhance time-delay estimation. We derive the corresponding Maximum Likelihood Estimator (MLE) and propose a practical implementation to evaluate its Mean Squared Error. Leveraging existing Cramér-Rao Bound results, we show that the MLE is efficient over a finite SNR range, though not asymptotically consistent or efficient.

***Index Terms*—** Cramér-Rao bound, time-delay and phase estimation, multi-sensor.

## 1. INTRODUCTION

The estimation of a received signal's time-delay provides insights into the propagation characteristics of the medium it traversed. Such information is exploited by a wide range of applications [1, 2, 3], and is particularly valuable in domains like Global Navigation Satellite Systems (GNSS) [4, 5], RADAR, SONAR [1, 6], or digital communications [7], where it acts as the first step in the receiver's operating sequence for synchronization purposes.

Traditional time-delay estimation models consider a single sensor link and often include a carrier-phase term $\phi = \varphi - w_c\tau$, where $\varphi$ captures unknown phase offsets (e.g., due to hardware misalignment), and $w_c = 2\pi F_c$ relates to the carrier frequency. A significant body of work has addressed this problem from a theoretical standpoint [8, 9, 10, 11, 12], deriving both Cramér-Rao Bounds (CRBs) and Maximum Likelihood Estimators (MLEs) under various modeling assumptions.

Two common approaches to the unknown phase term exist: one treats $\varphi$ as a nuisance parameter absorbed into the signal amplitude [13, 14], while the other assumes ideal calibration, allowing precise estimation of $w_c\tau$ [15, 16]. However, the latter assumption is often unrealistic in practical systems, where perfect calibration of the global phase is difficult to achieve. This study explores a multi-sensor scenario in which two distinct sensors are available: one dedicated to measuring the time-delay $\tau$, and another providing an auxiliary estimate of the global phase offset $\varphi$. This heterogenous sensing architecture is common in practical systems, where different physical modules (or subsystems) contribute complementary information. Our goal is to analyze how incorporating a partially informed measurement of $\varphi$, subject to known uncertainty, can enhance the performance of $\tau$ estimation.

Building on the CRB analysis introduced in [17], we formulate the corresponding MLE for this hybrid measurement model. The proposed estimator integrates both the received signal and the auxiliary phase information, and is compared against the CRB and a simpler estimator using raw phase estimates.

This contribution is organized as follows. Section 2 presents the signal model and sensor configuration. Section 3 derives the proposed MLE. Section 4 discusses performance results. Lastly, Section 5 summarizes the contents of the contribution and highlights its key findings.

## 2. SIGNAL MODEL

The signal model used in this study is based on the well-known Conditional Signal Model (CSM) [18, 19]. Specifically, we examine a signal model studied in the state of the art [16], where it is assumed that a band-limited signal at the output of the Hilbert filter can be expressed as,

$$x(t) = \alpha c(t-\tau)e^{j\varphi}e^{-jw_c\tau} + n(t),\ n(t) \sim \mathcal{CN}(0, \sigma_n^2), \quad (1)$$

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and the underlying vector of deterministic unknowns is,

$$\boldsymbol{\epsilon} = (\sigma_n^2, \alpha, \varphi, \tau)^T, \quad \boldsymbol{\zeta} = (\alpha, \varphi, \tau)^T, \quad \boldsymbol{\theta} = (\varphi, \tau)^T, \quad (2)$$

with $\sigma_n^2$ determining the noise variance and $\alpha > 0$ a real amplitude. For simplification, (1) is reformulated as,

$$x(t) = \alpha e^{j\varphi}\mathbf{a}(t;\tau) + n(t), \quad (3)$$

where $\mathbf{a}(t;\tau) = c(t-\tau)e^{-jw_c\tau}$. Note that in most of the state-of-the-art (e.g. [1, 7, 13]), it is assumed a signal model, where the carrier-phase term $\phi = \varphi - w_c\tau$ combines the phase component $\varphi$ with the wave propagation $w_c\tau$. In our case of study, it is assumed the availability of an additional data $x_a$ representing the output of a calibration step of the true phase offset $\varphi$, modeled as,

$$x_a = \varphi + n_a, \quad n_a \sim \mathcal{N}(0, \sigma_{n_a}^2), \quad (4)$$

where $\sigma_{n_a}^2$ denotes the known variance of the calibration step. A discrete-time signal representation can be formulated considering the acquisition of $N' = N_2' - N_1' + 1$ samples from (3). Assuming, $T_s = 1/F_s$ and $F_s \geq B$, where $T_s$ is the sampling period, $F_s$ the sampling rate, and $B$ the signal's bandwidth, the following signal model is established,

$$\mathbf{x} = \alpha e^{j\varphi}\mathbf{a}(\tau) + \mathbf{n}, \quad (5)$$

with $\mathbf{x}^T = (x(N_1'T_s), \cdots, x(N_2'T_s))$ the received signal samples, $\mathbf{n}^T = (n(N_1'T_s), \cdots, n(N_2'T_s))$ and $\mathbf{a}(\tau) = e^{-jw_c\tau}\mathbf{c}(\tau)$, $\mathbf{c}^T(\tau) = (c(N_1'T_s - \tau), \cdots, c(N_2'T_s - \tau))$.

## 3. MAXIMUM LIKELIHOOD ESTIMATORS

This section aims to introduce three estimators. The first is the MLE considering a signal model with perfect phase compensation. In other words, we assume that the phase estimated by the calibration step is the true phase offset, i.e., $x_a = \varphi$. That estimator was first derived in [16] and is expressed as,

$$\widehat{\tau}_1 = \underset{\left\{\tau | \Re\left\{\mathbf{a}(\tau)^H\left(e^{-j\varphi}\mathbf{x}\right)\right\}>0\right\}}{\arg\max} \left\{ \Re\left\{ \frac{\mathbf{a}^H(\tau)\left(e^{-j\varphi}\mathbf{x}\right)}{\|\mathbf{a}(\tau)\|} \right\}^2 \right\}, \quad (6)$$

where $\Re\{\cdot\}$ defines the real part operator. The second estimator, which can be regarded as a naive estimator, considers the signal model introduced in (3) and (4), but ignores the fact that $x_a$ is a random variable and assumes that $x_a = \varphi$. Note that, from a theoretical viewpoint, that estimator is a misspecified MLE (MMLE) [20]. That estimator is given by

$$\widehat{\tau}_2 = \underset{\{\tau | \Re\{\gamma(\tau)>0\}}{\arg\max} \left\{ \Re\left\{\gamma(\tau)\right\}^2 \right\}, \quad (7)$$

$$\gamma(\tau) = \frac{\mathbf{a}(\tau)^H\left(e^{-jx_a}\mathbf{x}\right)}{\|\mathbf{a}(\tau)\|}, \quad (8)$$

where $\gamma(\tau)$ consists in the normalized auto-correlation function of $\mathbf{a}(\tau)$ weighted by $e^{-jx_a}$ (unknown phase naive compensation). Finally, the MLE for the signal model proposed in (3) and (4) is given by

$$\widehat{\boldsymbol{\epsilon}} = \arg\max_{\boldsymbol{\epsilon}}\{p(\mathbf{x}, x_a; \boldsymbol{\epsilon}) = p(\mathbf{x}; \boldsymbol{\epsilon})p(x_a; \varphi)\}, \quad (9)$$

and the underlying negative log-likelihood function is

$$\mathcal{S}_0^a(\mathbf{x}; \boldsymbol{\epsilon}) = -\ln p(\mathbf{x}; \boldsymbol{\epsilon}) - \ln p(x_a; \varphi). \quad (10)$$

Hence, the MLE in (9) can be reformulated as,

$$\widehat{\boldsymbol{\epsilon}} = \arg\min_{\boldsymbol{\epsilon}}\{\mathcal{S}_0^a(\mathbf{x}; \boldsymbol{\epsilon})\}. \quad (11)$$

According to (10), the minimization in (11) w.r.t $\alpha$ and $\sigma_n^2$ takes only into account $\ln p(\mathbf{x}; \boldsymbol{\epsilon})$. This problem was already solved in [16, §3], where the reader can find the MLE expressions for $\widehat{\sigma^2}_n$ and $\hat{\alpha}$. Thus, following a similar methodology, the MLE expression for $\boldsymbol{\theta}$ taking into account (9) is given by

$$\widehat{\boldsymbol{\theta}} = \arg \min_{\{\boldsymbol{\theta} | \mathrm{Re}\{e^{-j\varphi}\mathbf{a}(\tau)^{\mathrm{H}}\mathbf{x}\}>0\}} \{\mathcal{S}_2^a(\mathbf{x}; \boldsymbol{\theta})\}, \quad (12)$$

$$\mathcal{S}_2^a(\mathbf{x}; \boldsymbol{\theta}) = \frac{(x_a - \varphi)^2}{2N\sigma_{n_a}^2} + \ln(\|\mathbf{x}\|^2 - \Re\left\{e^{j(x_a-\varphi)}\gamma(\tau)\right\}^2). \quad (13)$$

Moreover, under the assumption that $\sigma_n^2$ is known or accurately estimated, (13) becomes

$$\mathcal{S}_2^a(\mathbf{x}; \boldsymbol{\theta}) = \frac{\sigma_n^2}{2\sigma_{n_a}^2}(x_a - \varphi)^2 - \Re\left\{e^{j(x_a-\varphi)}\gamma(\tau)\right\}^2. \quad (14)$$

For the sake of simplicity and space, in this communication, we focus on that specific case, leaving the unknown $\sigma_n^2$ case (13) for future studies. Let us consider the following variable change $\varphi = \psi + x_a$, then $\mathcal{S}_2^a(\mathbf{x}; \boldsymbol{\theta})$ in (14) becomes

$$\mathcal{S}_2^a(\mathbf{x}; \boldsymbol{\theta}) = \frac{\sigma_n^2}{2\sigma_{n_a}^2}\psi^2 - \Re\left\{e^{-j\psi}\gamma(\tau)\right\}^2. \quad (15)$$

Interestingly enough, setting $\psi = 0$ in (15) recast (12) as (7); thus, the naive estimator amounts to a default choice of $\varphi$ ($\varphi = x_a$) if no prior pdf is known. Furthermore, if $z$ is a complex number, then $\Re\{z\}^2 = \left(\Re\{z^2\} + |z|^2\right)/2$. Thus, the second right term in (15) can be recast as

$$\begin{aligned}\Re\left\{e^{-j\psi}\gamma(\tau)\right\}^2 &= \frac{\Re\left\{\left(e^{-j\psi}\gamma(\tau)\right)^2\right\} + \left|e^{-j\psi}\gamma(\tau)\right|^2}{2} \\ &= \frac{\Re\left\{e^{-j2\psi}\gamma(\tau)^2\right\} + |\gamma(\tau)|^2}{2}. \quad (16)\end{aligned}$$

Assuming an operating range of the MLE in asymptotic regime, i.e. $\sigma_{n_a}^2 \ll 1 \Rightarrow \psi \ll 1$, $e^{-j2\psi}$ can be approximated by a fifth-order Taylor expansion

$$e^{-j2\psi} \simeq 1 + (-j2\psi) + \frac{1}{2}(-j2\psi)^2 + \frac{1}{6}(-j2\psi)^3 + \frac{1}{24}(-j2\psi)^4 + \frac{1}{120}(-j2\psi)^5 \simeq 1 - 2\psi^2 + \frac{2}{3}\psi^4 - j\left(2\psi - \frac{4}{3}\psi^3 + \frac{4}{15}\psi^5\right). \quad (17)$$

Hence, (16) becomes,

$$\Re\left\{e^{-j\psi}\gamma(\tau)\right\}^2 \simeq \frac{2}{15}\,\mathrm{Im}\left\{\gamma(\tau)^2\right\}\psi^5 \; \frac{1}{3}\Re\left\{\gamma(\tau)^2\right\}\psi^4 - \frac{2}{3}\,\mathrm{Im}\left\{\gamma(\tau)^2\right\}\psi^3 - \Re\left\{\gamma(\tau)^2\right\}\psi^2 + \mathrm{Im}\left\{\gamma(\tau)^2\right\}\psi + \Re\left\{\gamma(\tau)\right\}^2, \quad (18)$$

with $\mathrm{Im}\{\cdot\}$ defines the imaginary part operator, and as a result, (15) is recast as,

$$\mathcal{S}_2^a(\mathbf{x};\boldsymbol{\theta}) \simeq -\frac{2}{15}\,\mathrm{Im}\left\{\gamma(\tau)^2\right\}\psi^5 - \frac{1}{3}\Re\left\{\gamma(\tau)^2\right\}\psi^4 + \frac{2}{3}\,\mathrm{Im}\left\{\gamma(\tau)^2\right\}\psi^3 + \frac{1}{2}\left(\frac{\sigma_n^2}{\sigma_{n_a}^2} + 2\Re\left\{\gamma(\tau)^2\right\}\right)\psi^2 - \mathrm{Im}\left\{\gamma(\tau)^2\right\}\psi - \Re\left\{\gamma(\tau)\right\}^2. \quad (19)$$

With this redefinition of (15), the MLE estimator $\widehat{\psi}$ is expressed as a function of $\tau$ as

$$\widehat{\psi}(\tau) = \arg\min_{\{\psi|\Re\{e^{-j\psi}\gamma(\tau)\}>0\}}\left\{\mathcal{S}_2^a(\mathbf{x};\psi,\tau)\right\} = \arg_{\{\psi|\Re\{e^{-j\psi}\gamma(\tau)\}>0\}}\left\{\frac{\partial\mathcal{S}_2^a(\mathbf{x};\psi,\tau)}{\partial\psi} = 0\right\}. \quad (20)$$

with $\partial\mathcal{S}_2^a(\mathbf{x};\psi,\tau)/\partial\psi = 0$ being solvable from Ferrari's method since

$$\frac{\partial\mathcal{S}_2^a(\mathbf{x};\psi,\tau)}{\partial\psi} \simeq -\frac{2}{3}\,\mathrm{Im}\left\{\gamma(\tau)^2\right\}\psi^4 - \frac{4}{3}\Re\left\{\gamma(\tau)^2\right\}\psi^3 + 2\,\mathrm{Im}\left\{\gamma(\tau)^2\right\}\psi^2 - \mathrm{Im}\left\{\gamma(\tau)^2\right\} + \left(\frac{\sigma_n^2}{\sigma_{n_a}^2} + 2\Re\left\{\gamma(\tau)^2\right\}\right)\psi, \quad (21)$$

is a polynomial of forth degree. Note that in this particular case, Ferrari's method yields four possible solutions of $\psi$, each corresponding to one of the roots of the polynomial in (21). Moreover, since $\psi$ is required to be real valued, we consider only the real part of the obtained roots and the selected root of $\widehat{\psi} \equiv \widehat{\psi}(\tau)$ is the one which meets the criteria:

$$\widehat{\psi} = \arg\min\{\mathcal{S}_2^a\left(\mathbf{x};\Re\{\widehat{\psi}_1\},\tau\right),\ldots,\mathcal{S}_2^a\left(\mathbf{x};\Re\{\widehat{\psi}_4\},\tau\right)\}. \quad (22)$$

In other words, we search for the real part of the roots of (20) that minimizes the cost function (15). As a result, the MLE $\widehat{\tau}$ becomes,

$$\widehat{\tau}_3 = \arg\max_{\tau}\left\{\mathcal{S}_2^a\left(\mathbf{x};\widehat{\psi}(\tau),\tau\right)\right\}$$

$$\widehat{\tau}_3 = \arg\max_{\tau}\left\{\frac{\sigma_n^2\widehat{\psi}(\tau)^2}{2\sigma_{n_a}^2} - \Re\left\{e^{-j\widehat{\psi}(\tau)}\gamma(\tau)\right\}^2\right\}. \quad (23)$$

## 4. VALIDATION OF THE RESULTS

This section validates the theoretical framework described in section 3 and analyzes the results obtained after software implementation. The simulations conducted consider the signal model introduced in (1) and (4), where $c(t)$ is a GPS L1 C/A [21] periodic Gold sequence of 1,023 chips, modulated by Binary Phase Shift Keying (BPSK) with a carrier frequency $F_c$ of 1,575.42 MHz. The sampling frequency $F_s$ is set to 4 MHz, the integration time is of 1 ms and the number of Monte Carlo runs is set to 1000. The SNR at the output of the MLE, often referred to as the matched filter [22], is defined for the true time-delay parameter $\tau^0$ as:

$$\mathrm{SNR}_{OUT} = \left.\frac{\mathrm{Re}\left\{\left(\frac{\mathbf{a}(\tau)}{\|\mathbf{a}(\tau)\|}\right)^{\mathrm{H}}\left(\alpha^0\mathbf{a}(\tau^0)\right)\right\}^2}{\mathrm{E}\left[\mathrm{Re}\left\{\left(\frac{\mathbf{a}(\tau)}{\|\mathbf{a}(\tau)\|}\right)^{\mathrm{H}}\mathbf{n}\right\}^2\right]}\right|_{\tau=\tau^0} = \frac{\left(\alpha^0\right)^2\left\|\mathbf{a}(\tau^0)\right\|^2}{\frac{(\sigma_n^0)^2}{2}} = \frac{2\left\|\mathbf{a}(\tau^0)\right\|^2}{\left(\sigma_n^0\right)^2}\left(\alpha^0\right)^2. \quad (24)$$

Two separate tests were conducted, resulting in Figures 1 and 2. Both Figures provides the Root Mean Square Error (RMSE) of the time-delay estimate as a function of the $\mathrm{SNR}_{OUT}$. Moreover, both Figures include the asymptotic performance provided by three CRB expressions derived and validated in previous state of the art. i) $\mathrm{CRB}^{\mathrm{m}}$ from [13], which considers a CSM where the $\varphi$ term is incorporated into $\alpha$ and corresponds to the classic signal model approach. Note that this signal model assumes that $\sigma_{n_a}^2 \to \infty$, meaning that $x_a$ provides no useful information at the system, making it preferable not to use it. ii) $\mathrm{CRB}^{\mathrm{e}}$ from [16], which assumes a CSM with perfect $\varphi$ compensation. In other words, $\sigma_{n_a}^2 \to 0$ and an ideal knowledge of $\varphi$ is assumed. iii) $\mathrm{CRB}^{\mathrm{f}}$ from [17], which assumes the CSM introduced in (3) and (4) and which will allow us to validate the MLE $\widehat{\tau}_3$ derived in equation (23). $\mathrm{CRB}^{\mathrm{f}}$ is expressed as follows [17]:

$$CRB^f = \frac{\sigma_n^2}{2\alpha^2}\left(\begin{array}{c}\Re\left\{\left(\frac{\partial\mathbf{c}(\tau)}{\partial\tau}\right)^H\mathbf{\Pi}^{\perp}_{\mathbf{c}(\tau)}\frac{\partial\mathbf{c}(\tau)}{\partial\tau}\right\} + \\ \frac{\|\mathbf{c}(\tau)\|^2\left(2\pi f_c - \frac{\mathrm{Im}\left\{\mathbf{c}(\tau)^H\frac{\partial\mathbf{c}(\tau)}{\partial\tau}\right\}}{\|\mathbf{c}(\tau)\|^2}\right)^2}{1+2\sigma_{n_a}^2 SNR_{out}}\end{array}\right)^{-1} \quad (25)$$

where $\mathbf{\Pi}_{\mathbf{c}(\tau)}^{\perp} = \mathbf{I} - \mathbf{c}(\tau)\mathbf{c}(\tau)^H / \|\mathbf{c}(\tau)\|^2$.

Figure 1 serves two primary purposes. The first one is to validate the closed-form $\mathrm{CRB}^{\mathrm{f}}$ expression (25) derived in [17]. The second one is to check the proposed implementation of the MLE $\widehat{\tau}_3$ (23) for the CSM in (3) and (4). It is important

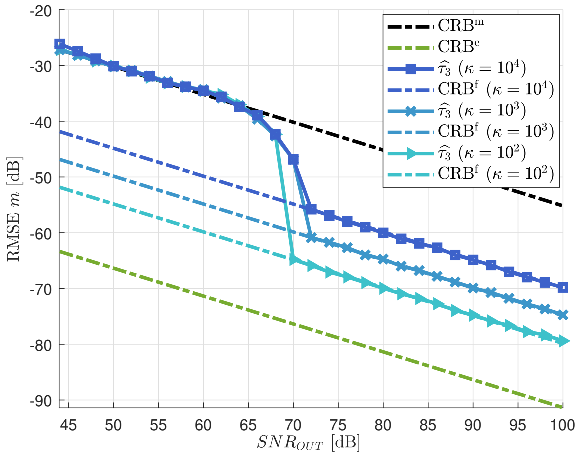


**Fig. 1**. RMSE of $\widehat{\tau}_3$ as a function of $\mathrm{SNR}_{OUT}$ for $\kappa = \left[10^2, 10^3, 10^4\right]$.

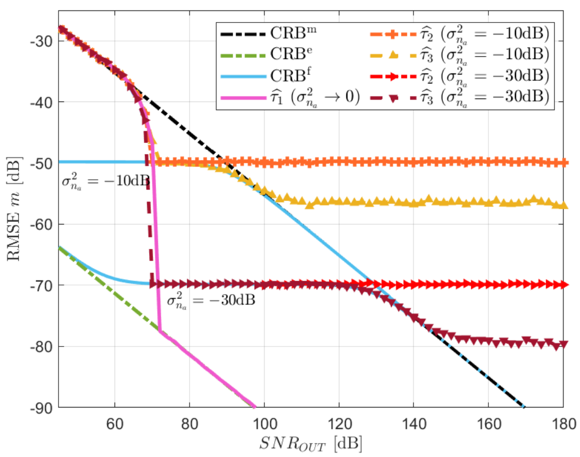


**Fig. 2**. RMSE of $\widehat{\tau}_1$, $\widehat{\tau}_2$ and $\widehat{\tau}_3$ as a function of $\mathrm{SNR}_{OUT}$. For $\widehat{\tau}_2$ and $\widehat{\tau}_3$, $\sigma_{n_a}^2 = [-10, -30]$dB.

to remind that under the CSM condition [1, 19], the MSE of the MLE asymptotically converges to the CRB, enabling validation of both the CRB expressions and the MLE implementation. However, the asymptotic regime of the CSM discussed in this paper (as defined by (3) and (4)) implies that not only $\mathrm{SNR}_{OUT} \to \infty$, but $\sigma_{n_a}^2$ also approaches zero. Therefore, let us set the constant $\kappa$ such as $\sigma_{n_a}^2 = \kappa / SNR_{out}$, where $\kappa$ represents the extent to which $\sigma_{n_a}^2$ decreases as the SNR increases. Under these conditions, Figure 1 shows that the MSE of the MLE converges to the corresponding $CRB^f$ for several scenarios with $\kappa = \left[10^2, 10^3, 10^4\right]$. Figure 2 shows the asymptotic behavior of the three estimators $\widehat{\tau}_1$ in (6), $\widehat{\tau}_2$ in (7) and $\widehat{\tau}_3$ in (23). On the one hand, $\widehat{\tau}_1$ estimator is simulated under the condition of perfect phase compensation, i.e., with $\sigma_{n_a}^2 \to 0$. Note that under these conditions, the MSE of $\widehat{\tau}_1$, that is the MLE, converges asymptotically to $\mathrm{CRB}^{\mathrm{e}}$ derived in [16]. On the other side, $\widehat{\tau}_2$ and $\widehat{\tau}_3$ estimators are simulated under a scenario where the SNR increases but the phase term variance is set to two fixed values $\sigma_{n_a}^2 = [-10, -30]$dB. Surprisingly, estimators $\widehat{\tau}_2$ and $\widehat{\tau}_3$ are efficient in certain regions of SNR, i.e. the MSE of both estimators converges to $\mathrm{CRB}^{\mathrm{f}}$, but from certain respective SNR value, they become neither efficient nor even consistent. A sensible explanation is that they do not operate in the asymptotic conditions for the considered CSM ($\mathrm{SNR}_{OUT} \to \infty$, and $\sigma_{n_a}^2 \to 0$). Note also that the asymptotic MSE associated with each estimator is proportional to $\sigma_{n_a}^2$, which is logical, as the estimators asymptotic performance is primarily determined by the precision of the estimated phase term (asymptotically, the contribution of the additive noise $\boldsymbol{n}$ is negligible). Moreover, as expected, the addition of a prior knowledge on the unknown phase decreases the achievable MSE of $\widehat{\tau}_3$ (a well known result in Bayesian framework [23]). Last, we want to highlight the significance of the $\mathrm{CRB}^{\mathrm{m}}$ and $\mathrm{CRB}^{\mathrm{e}}$ bounds, as they provide not only a lower bound on system performance in the best-case scenario for high SNRs, but also an upper bound for system performance in intermediate SNR regions (which may be in the order of [15-60] dB of $\mathrm{SNR}_{OUT}$ as shown in [16]).

## 5. CONCLUSION

This work introduces new insights into time-delay estimation theory in the context of multi-sensor systems. We focus on a recently proposed CSM in which the estimation of the time-delay parameter is assisted by phase estimates obtained from an auxiliary calibration sensor. Based on this model, we propose two estimators for the time-delay. The first, denoted $\widehat{\tau}_2$, assumes perfectly accurate phase estimates. While simple to implement, this naive approach suffers from model mismatch and is therefore asymptotically suboptimal. The second estimator, $\widehat{\tau}_3$, corresponds to the Maximum Likelihood Estimator (MLE) under the proposed CSM, and fully accounts for the uncertainty in the phase calibration. To validate the theoretical derivation, we analyze the MSE of $\widehat{\tau}_3$ and show that it converges to the CRB in the asymptotic regime, i.e., when the SNR tends to infinity and the phase estimate variance tends to zero. Furthermore, we examine a more realistic scenario where the variance of the phase estimates is fixed. In this case, the MSE of $\widehat{\tau}_3$ approaches the CRB only within a specific SNR range. As the SNR increases, the estimator becomes neither efficient nor consistent, and its asymptotic performance is ultimately limited by the quality of the phase sensor.

## 6. REFERENCES


[1] H. L. Van Trees, *Optimum Array Processing*, Wiley-Interscience, New-York, 2002.

[2] Jingdong Chen, Yiteng Huang, and Jacob Benesty, "Time delay estimation," *Audio signal processing for next-generation multimedia communication systems*, pp. 197–227, 2004.

[3] Bernard C Levy, *Principles of signal detection and parameter estimation*, Springer Science & Business Media, 2008.

[4] E. D. Kaplan, Ed., *Understanding GPS: principles and applications*, Artech House, 2nd edition, 2006.

[5] P. J. G. Teunissen and O. Montenbruck, Eds., *Handbook of Global Navigation Satellite Systems*, Springer, Switzerland, 2017.

[6] D. A. Swick, "A Review of Wideband Ambiguity Functions," Tech. Rep. 6994, Naval Res. Lab., Washington DC, 1969.

[7] U. Mengali and A. N. D'Andrea, *Synchronization Techniques for Digital Receivers*, Plenum Press, New York, USA, 1997.

[8] H. L. Van Trees, *Detection, Estimation, and Modulation Theory, Part III: Radar – Sonar Signal Processing and Gaussian Signals in Noise*, J. Wiley & Sons, 2001.

[9] Q. Jin, K. M. Wong, and Z.-Q. Luo, "The Estimation of Time Delay and Doppler Stretch of Wideband Signals," *IEEE Trans. Signal Process.*, vol. 43, no. 4, pp. 904–916, April 1995.

[10] A. Dogandzic and A. Nehorai, "Cramér-Rao bounds for estimating range, velocity, and direction with an active array," *IEEE Trans. Signal Process.*, vol. 49, no. 6, pp. 1122–1137, June 2001.

[11] N. Noels, H. Wymeersch, H. Steendam, and M. Moeneclaey, "True Cramér-Rao bound for timing recovery from a bandlimited linearly modulated waveform with unknown carrier phase and frequency," *IEEE Trans. on Communications*, vol. 52, no. 3, pp. 473–483, March 2004.

[12] Y. Shangfu W. He-Wen and W. Qun, "Influence of random carrier phase on true cramér-rao lower bound for time delay estimation," in *Proc. of the IEEE Intl. Conf. on Acoustics, Speech and Signal Processing (ICASSP)*, Honolulu, USA, April 2007.

[13] D. Medina, L. Ortega, J. Vilà-Valls, P. Closas, François Vincent, and E. Chaumette, "Compact CRB for Delay, Doppler, and Phase Estimation – Application to GNSS SPP and RTK Performance Characterisation," *IET Radar, Sonar & Navigation*, vol. 14, no. 10, pp. 1537–1549, 2020.

[14] P. Das, J. Vilà-Valls, F. Vincent, L. Davain, and E. Chaumette, "A New Compact Delay, Doppler Stretch and Phase Estimation CRB with a Band-Limited Signal for GenE. Remote Sensing Applications," *Remote Sensing*, vol. 12, no. 18, pp. 2913, Sep. 2020.

[15] J.M. Bernabeu, L. Ortega, A. Blais, Y. Gregoire, and E. Chaumette, "Time-delay and doppler estimation with a carrier modulated by a band-limited signal," in *2023 IEEE 9th International Workshop on Computational Advances in Multi-Sensor Adaptive Processing (CAMSAP)*, 2023, pp. 346–350.

[16] J.M. Bernabeu, L. Ortega, A. Blais, Y. Gregoire, and E. Chaumette, "On the asymptotic performance of time-delay and doppler estimation with a carrier modulated by a band-limited signal," *EURASIP - JASP*, 2023, Article ID 295029. DOI: 10.21203/rs.3.rs-3539143/v1.

[17] J.M. Bernabeu, L. Ortega, A. Blais, Y. Gregoire, and E. Chaumette, "On time-delay estimation accuracy limit under phase uncertainty," in *2024 27th International Conference on Information Fusion (FUSION), (Accepted)*, 2024.

[18] P. Stoica and A. Nehorai, "Performances study of conditional and unconditional direction of arrival estimation," *IEEE Trans. Acoust., Speech, Signal Process.*, vol. 38, no. 10, pp. 1783–1795, Oct. 1990.

[19] A. Renaux, P. Forster, E. Chaumette, and P. Larzabal, "On the high-SNR conditional maximum-likelihood estimator full statistical characterization," *IEEE Trans. Signal Process.*, vol. 54, no. 12, pp. 4840 – 4843, Dec. 2006.

[20] H. McPhee, L. Ortega, J. Vilà-Valls, and E. Chaumette, "On the accuracy limits of misspecified delay-doppler estimation," *Signal Processing*, vol. 205, pp. 108872, 2023.

[21] Michael J Dunn and D Disl, "Global positioning system directorate systems engineering & integration interface specification is-gps-200," *Global Positioning Systems Directorate*, 2012.

[22] S. M. Kay, *Fundamentals of Statistical Signal Processing: Estimation Theory*, Prentice-Hall, Englewood Cliffs, New Jersey, USA, 1993.

[23] Harry L Van Trees and Kristine L Bell, "Bayesian bounds for parameter estimation and nonlinear filtering/tracking," *AMC*, vol. 10, pp. 12, 2007.