ON THE EFFICIENCY OF A MISSPECIFIED CONTAMINATED NONLINEAR REGRESSION MODEL: APPLICATION TO TIME-DELAY AND DOPPLER ESTIMATION

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Hamish McPhee [1], *Lorenzo Ortega* [1,2], *Stefano Fortunati*[3], *Jean-Yves Tourneret* [1,4]

[1] TéSA, [2] IPSA, [3], Institut Polytechnique de Paris/SAMOVAR, [4] IRIT-ENSEEIHT, University of Toulouse

ABSTRACT

Nonlinear regression models play a crucial role in signal processing and multi-sensor applications. Traditionally, performance bounds for these models assume independent Gaussian observations. In practice, the Gaussian assumption fails in multi-sensor systems if some proportion of sensors are corrupted by non-Gaussian noise and outliers. In this context, we extend the Misspecified Cramér-Rao Bound (MCRB) framework to the contaminated Gaussian noise model, where observations are generated from a mixture of nominal Gaussian noise and occasional outliers. Building on previous work with Complex Elliptically Symmetric noise models, we derive analytical MCRB expressions under the mismatched Gaussian assumption and study the asymptotic behavior of the corresponding Misspecified Maximum Likelihood Estimator (MMLE). To demonstrate practical relevance, we apply the theory to joint time-delay and Doppler estimation in GPS signals under contamination. Numerical simulations confirm that the MMLE root mean squared error converges to the theoretical MCRB, which aligns with the classical Gaussian CRB.

Index Terms— Cramér-Rao bound, time-delay and Doppler estimation, band-limited signals.

1. INTRODUCTION

Nonlinear regression models are ubiquitous in signal processing and multi-sensor applications, ranging from remote sensing to radar and navigation [1–10]. Traditionally, the derivation of performance bounds and estimators in such models relies on the assumption that the observed data are independent and identically distributed (i.i.d.) and follow a Gaussian distribution [11–13]. While analytically convenient, this assumption often fails to capture the complexity of real-world scenarios, where the data may exhibit non-Gaussian characteristics such as heavy tails or contamination by outliers.

Recently, the Misspecified Cramér-Rao Bound (MCRB) [14,15] has emerged as a powerful tool to assess the performance of estimators derived under mismatched statistical assumptions. In particular, when the true data-generating process deviates from the assumed Gaussian model, the MCRB provides a meaningful lower bound on the Mean Squared Error (MSE) of the Misspecified Maximum Likelihood Estimator (MMLE). The MMLE is the estimator that maximizes the likelihood of a model that is not the true signal model. This paper builds upon previous work on the MCRB for nonlinear regression by extending the analysis of one of the most important non-Gaussian noise models: the contaminated Gaussian noise, where the observations arise from a mixture of nominal Gaussian noise and occasional outliers. While our earlier study focused on noise with Complex Elliptically Symmetric (CES) distributions [16], this study explores a more specific contamination model to assess its impact on estimation bounds and validate the robustness of the MCRB under realistic noise conditions.

Contributions of this work include: derivation of the corresponding MCRB expressions under the mismatched Gaussian assumption and analysis of the associated MMLE. We show the impact that contamination has on the estimation performance by assessing the asymptotic properties of the MMLE in these non-ideal conditions. The verified closed-form MCRB then allows us to more simply understand the losses in the presence of contamination without needing to run extensive computational simulations. To illustrate the practical implications of our findings, we apply the developed theory to the problem of time-delay and Doppler shift estimation in Global Navigation Satellite Systems (GNSS), a representative multi-sensor application where robustness to non-Gaussian noise is critical.

This work was supported by TéSA laboratory, Toulouse, France.

2. SIGNAL MODEL

2.1. True signal model

Consider a complex-valued vector $\boldsymbol{x} = (x_1, ..., x_K)^\top \in \mathbb{C}^K$ defined by the following data generating process:

$$\mathbf{x} = \mathbf{f}(\bar{\boldsymbol{\theta}}) + \mathbf{n}, \tag{1}$$

where $\mathbf{n} = (n_1, ..., n_K)^\top \in \mathbb{C}^K$ is a *zero-mean* complex-valued noise vector whose K entries are assumed to be independent and identically distributed (i.i.d.), $x_k = f_k(\bar{\boldsymbol{\theta}}) + n_k, k = 1, ..., K$, $\bar{\boldsymbol{\theta}} \in \Theta \subset \mathbb{R}^p$ indicates the real-valued *true parameter vector*, Θ is a compact subset of $\mathbb{R}^p$. The functions $f_k : \Theta \to \mathbb{C},\ k \in \mathbb{Z}$ are supposed to be *known continuous and differentiable* functions defined on Θ. The noise vector $\mathbf{n}$ has independent components distributed according to a bimodal Gaussian mixture distribution modeling the potential presence of outliers. This model has been used for outliers in several references including [17–19]. It is defined by the proportion of contaminated data $\bar{\epsilon}$ and the variance scaling factor $\bar{\kappa}$ such that $n_k \sim (1 - \bar{\epsilon})\mathcal{CN}(0, \bar{\sigma}_n^2) + \bar{\epsilon}\mathcal{CN}(0, \bar{\kappa}\bar{\sigma}_n^2)$, where $\mathcal{CN}(0, \bar{\sigma}_n^2)$ denotes the complex Gaussian distribution with mean 0 and variance $\bar{\sigma}_n^2$. The distribution of the noise vector in (1) yields:

$$p_{\bar{\boldsymbol{\zeta}}}(\mathbf{x}, \bar{\boldsymbol{\zeta}}) = (1 - \bar{\epsilon})\mathcal{CN}(\mathbf{f}(\bar{\boldsymbol{\theta}}), \bar{\sigma}_n^2 \boldsymbol{I}_n) + \bar{\epsilon}\mathcal{CN}(\mathbf{f}(\bar{\boldsymbol{\theta}}), \bar{\kappa}\bar{\sigma}_n^2 \boldsymbol{I}_n), \tag{2}$$

which depends on the parameter vector $\bar{\boldsymbol{\zeta}}^\top = \left(\bar{\kappa}, \bar{\epsilon}, \bar{\sigma}_n^2, \bar{\boldsymbol{\theta}}^\top\right)$.

2.2. Misspecified Gaussian i.i.d. signal model

To estimate the parameter vector $\bar{\boldsymbol{\theta}}$, practitioners often assume a simplified statistical model instead of the true data-generating process in (1). This model misspecification is mainly due to: (i) the unknown and hard-to-characterize noise structure; and (ii) the need for tractable and efficient estimation algorithms [14]. A common assumption is that the noise vector $\boldsymbol{n} \sim \mathcal{CN}(\mathbf{0}, \sigma_n^2 \boldsymbol{I}_K)$, leading to the following probability density function (pdf) for $\mathbf{x}$ in (1):

$$f_{\boldsymbol{\varphi}}(\mathbf{x}; \sigma_n^2, \boldsymbol{\theta}) = (\pi\sigma_n^2)^{-N} \exp\left(-\frac{||\mathbf{x} - \mathbf{f}(\boldsymbol{\theta})||^2}{\sigma_n^2}\right), \quad (3)$$

with $\boldsymbol{\varphi}^\top = (\sigma_n^2, \boldsymbol{\theta}^\top)$. The key question addressed next is: *can we derive a lower bound on the MSE of any unbiased (or consistent) estimator of $\bar{\boldsymbol{\theta}}$ under the misspecified Gaussian i.i.d. model, when the true signal is defined in* (1)? To answer this question, we evaluate the MCRB [14, 15, 20] for $\bar{\boldsymbol{\theta}}$ under the assumed model in (3), while the true process follows (1).

3. THE PSEUDO-TRUE PARAMETER VECTOR

The pseudo-true parameter vector $\boldsymbol{\varphi}_0$ is the element that minimizes the Kullback-Leibler Divergence (KLD) between the true pdf and any element $f_{\boldsymbol{\varphi}}$ of the misspecified model [14,15,21]. The KLD is defined as:

$$D(p_{\bar{\boldsymbol{\zeta}}}||f_{\boldsymbol{\varphi}}) = E_{p_{\bar{\boldsymbol{\zeta}}}}\left[\ln\left(\frac{p_{\bar{\boldsymbol{\zeta}}}(\mathbf{x}; \bar{\boldsymbol{\zeta}})}{f_{\boldsymbol{\varphi}}(\mathbf{x}; \boldsymbol{\varphi})}\right)\right], \mathbf{x} \sim p_{\bar{\boldsymbol{\zeta}}}, \quad (4)$$

where $E_{p_{\bar{\boldsymbol{\zeta}}}}[\cdot]$ is the expectation with respect to (w.r.t.) the true model pdf. Consequently:

$$\boldsymbol{\varphi}_0 = \arg\min_{\boldsymbol{\varphi}} \left\{D(p_{\bar{\boldsymbol{\zeta}}}||f_{\boldsymbol{\varphi}})\right\} = \arg\min_{\boldsymbol{\varphi}} \left\{E_{p_{\bar{\boldsymbol{\zeta}}}}\left[-\ln f_{\boldsymbol{\varphi}}(\mathbf{x}; \boldsymbol{\varphi})\right]\right\}. \quad (5)$$

From (3), it follows directly that:

$$\boldsymbol{\varphi}_0 = \arg\min_{\boldsymbol{\varphi}} \left\{E_{p_{\bar{\boldsymbol{\zeta}}}}\left[\frac{1}{\sigma_n^2}\left[\|\mathbf{x} - \mathbf{f}(\boldsymbol{\theta})\|^2\right]\right] + N\ln(\sigma_n^2)\right\} \quad (6)$$

Following [16], we first minimize (6) w.r.t $\boldsymbol{\theta}$:

$$\boldsymbol{\theta}_0 = \arg\min_{\boldsymbol{\theta}} \left\{E_{p_{\bar{\boldsymbol{\zeta}}}}\left[\|\mathbf{x} - \mathbf{f}(\boldsymbol{\theta})\|^2\right]\right\} = \arg\min_{\boldsymbol{\theta}} \left\{||\mathbf{f}(\bar{\boldsymbol{\theta}}) - \mathbf{f}(\boldsymbol{\theta})||^2\right\}. \quad (7)$$

leading to $\boldsymbol{\theta}_0 = \bar{\boldsymbol{\theta}}$. By using (7), the minimization of (6) with $\boldsymbol{\theta}_0 = \bar{\boldsymbol{\theta}}$ w.r.t. σ_n^2 yields:

$$\sigma_0^2 = \arg\min_{\sigma_n^2} \left\{E_{p_{\bar{\boldsymbol{\zeta}}}}\left[-\ln f_{\boldsymbol{\varphi}}(\mathbf{x}; \sigma_n^2, \bar{\boldsymbol{\theta}})\right]\right\}. \quad (8)$$

Straightforward computations lead to:

$$E_{p_{\bar{\boldsymbol{\zeta}}}}\left[\frac{\partial}{\partial\sigma_n^2}\ln f_{\boldsymbol{\varphi}}(\mathbf{x}; \sigma_n^2, \boldsymbol{\theta})\bigg|_{\sigma_n^2=\sigma_0^2}\right] \quad (9)$$

$$= E_{p_{\bar{\boldsymbol{\zeta}}}}\left[-\frac{N}{\sigma_n^2} + \frac{1}{\sigma_n^4}\|\mathbf{x} - \mathbf{f}(\boldsymbol{\theta})\|^2\bigg|_{\sigma_n^2=\sigma_0^2}\right]$$

$$= -\frac{N}{\sigma_0^2} + \frac{N\bar{\sigma}_n^2(1 + (\bar{\kappa} - 1)\bar{\epsilon})}{\sigma_0^4}. \quad (10)$$

Setting this derivative to 0 allows the following result to be obtained:

$$\sigma_0^2 = \bar{\sigma}_n^2(1 + (\bar{\kappa} - 1)\bar{\epsilon}).$$

4. DERIVATION OF $\mathbf{MCRB}(\boldsymbol{\varphi}_0)$

The aim of this section is to provide the closed form expression of the MCRB for the estimation of $\bar{\boldsymbol{\varphi}} = (\bar{\sigma}_n^2, \bar{\boldsymbol{\theta}}^T)$ under the misspecified scenario discussed in Section 2.2. Following [21], [14, Theo. 1] and [15, Theo. 4.1] and exploiting the pseudo-true parameter vector derived in the previous section, the MCRB for the parameter vector $\boldsymbol{\varphi}$ at point $\boldsymbol{\varphi}_0$ is given by:

$$\mathbf{MCRB}(\boldsymbol{\varphi}_0) = \mathbf{A}(\boldsymbol{\varphi}_0)^{-1}\mathbf{B}(\boldsymbol{\varphi}_0)\mathbf{A}(\boldsymbol{\varphi}_0)^{-1}, \quad (11)$$

where:

$$[\mathbf{A}(\boldsymbol{\varphi}_0)]_{i,j} \triangleq \left[E_{p_{\bar{\boldsymbol{\zeta}}}}\left[\nabla_{\boldsymbol{\varphi}}\nabla_{\boldsymbol{\varphi}}^\top \ln f_{\boldsymbol{\varphi}}(\mathbf{x}; \boldsymbol{\varphi}_0)\right]\right]_{i,j}$$
$$= E_{p_{\bar{\boldsymbol{\zeta}}}}\left[\frac{\partial^2}{\partial_i\partial_j}\ln f_{\boldsymbol{\varphi}}(\mathbf{x}; \boldsymbol{\varphi})\bigg|_{\boldsymbol{\varphi}=\boldsymbol{\varphi}_0}\right], \quad (12)$$

$$[\mathbf{B}(\boldsymbol{\varphi}_0)]_{i,j} \triangleq \left[E_{p_{\bar{\boldsymbol{\zeta}}}}\left[\nabla_{\boldsymbol{\varphi}} \ln f_{\boldsymbol{\varphi}}(\mathbf{x}; \boldsymbol{\varphi}_0)\nabla_{\boldsymbol{\varphi}}^\top \ln f_{\boldsymbol{\varphi}}(\mathbf{x}; \boldsymbol{\varphi}_0)\right]\right]_{i,j}$$
$$= E_{p_{\bar{\boldsymbol{\zeta}}}}\left[\frac{\partial}{\partial_i}\ln f_{\boldsymbol{\varphi}}(\mathbf{x}; \boldsymbol{\varphi})\bigg|_{\boldsymbol{\varphi}=\boldsymbol{\varphi}_0}\frac{\partial}{\partial_j}\ln f_{\boldsymbol{\varphi}}(\mathbf{x}; \boldsymbol{\varphi})\bigg|_{\boldsymbol{\varphi}=\boldsymbol{\varphi}_0}\right]. \quad (13)$$

Using the derivations summarized in Appendix A, the matrices $\boldsymbol{A}(\boldsymbol{\varphi}_0)$ and $\mathbf{B}(\boldsymbol{\varphi}_0)$ can be expressed as:

$$\mathbf{A}(\boldsymbol{\varphi}_0) = \begin{pmatrix} -N/\sigma_0^4 & \mathbf{0}_{1\times 4} \\ \mathbf{0}_{4\times 1} & -\frac{2}{\sigma_0^2}\sum_{k=1}^K \mathrm{Re}\left\{\nabla_{\boldsymbol{\theta}}\bar{f}_k\nabla_{\boldsymbol{\theta}}^H\bar{f}_k\right\} \end{pmatrix} \quad (14)$$

$$\mathbf{B}(\boldsymbol{\varphi}_0) = \begin{pmatrix} \frac{E_{p_{\bar{\boldsymbol{\zeta}}}}\left[(\mathbf{n}^H\mathbf{n})^2\right] - \sigma_0^4 N^2}{\sigma_0^8} & \mathbf{0}_{1\times 4} \\ \mathbf{0}_{4\times 1} & \frac{2}{\sigma_0^2}\sum_{k=1}^K \mathrm{Re}\left\{\nabla_{\boldsymbol{\theta}}\bar{f}_k\nabla_{\boldsymbol{\theta}}^H\bar{f}_k\right\} \end{pmatrix}. \quad (15)$$

hence

$$\mathbf{MCRB}(\boldsymbol{\varphi}_0) = \mathbf{A}(\boldsymbol{\varphi}_0)^{-1}\mathbf{B}(\boldsymbol{\varphi}_0)\mathbf{A}(\boldsymbol{\varphi}_0)^{-1} \quad (16)$$
$$= \begin{pmatrix} \frac{E_{p_{\bar{\boldsymbol{\zeta}}}}\left[(\mathbf{n}^H\mathbf{n})^2\right] - \sigma_0^4 N^2}{N^2} & \mathbf{0}_{1\times 4} \\ \mathbf{0}_{4\times 1} & \frac{\sigma_0^2}{2}\left(\sum_{k=1}^K \mathrm{Re}\left\{\nabla_{\boldsymbol{\theta}}\bar{f}_k\nabla_{\boldsymbol{\theta}}^H\bar{f}_k\right\}\right)^{-1} \end{pmatrix}.$$

Several key points should be highlighted. From (16), we can see that the term related to the parameters of interest, $\mathbf{MCRB}(\boldsymbol{\theta}_0) = \mathbf{MCRB}(\bar{\boldsymbol{\theta}})$, depends on the true noise distribution, characterized by ϵ and κ, i.e., on the contaminated model. This phenomenon differs from previously studied cases. For instance, when the true noise distribution is a CES distribution, the MCRB does not depend on the true distribution [22]. For the model studied in this work, one has:

$$\mathbf{MCRB}^{-1}(\bar{\boldsymbol{\theta}}) = \frac{2}{\bar{\sigma}_n^2(1 + (\bar{\kappa} - 1)\bar{\epsilon})}\sum_{k=1}^K \mathrm{Re}\left\{\nabla_{\boldsymbol{\theta}}\bar{f}_k\nabla_{\boldsymbol{\theta}}^H\bar{f}_k\right\}. \quad (17)$$

Note that the MCRB equals the Gaussian CRB [11, 22], but with scaling of the variance by $(1 + (\bar{\kappa} - 1)\bar{\epsilon})$. Consequently, it simplifies to the Gaussian CRB when the contamination is absent, i.e., when $\kappa = 1$ or $\epsilon = 0$ [11]. Finally, since the pseudo-true parameters of interest coincide with the true ones, we can explicitly confirm that the MMLE is asymptotically unbiased, as is also shown for CES-type noise models [22].

5. APPLICATION TO TIME-DELAY AND DOPPLER ESTIMATION

A band-limited signal $a(t)$ with bandwidth B is transmitted with a carrier frequency f_c from a transmitter T at position $\boldsymbol{P}_T(t)$ to a receiver R at position $\boldsymbol{P}_R(t)$. The distance travelled by the transmitted signal is $c\tau_0(t) = \boldsymbol{P}_{TR} = \|\boldsymbol{P}_T(t-\tau_0(t)) - \boldsymbol{P}_R(t)\| \approx (\boldsymbol{P}_T - \boldsymbol{P}_R) + vt$, where v is the relative velocity between the transmitter and the receiver and c is the speed of light. The received discrete signal at the output of the Hilbert filter is built from K samples at the sampling period $T_s = 1/F_s = 1/B$ [13, 23]:

$$\mathbf{x} = \bar{\alpha}\boldsymbol{\mu}(\bar{\boldsymbol{\eta}}) + \mathbf{n} = \bar{\rho}e^{j\bar{\Phi}}\boldsymbol{\mu}(\bar{\boldsymbol{\eta}}) + \mathbf{n}, \tag{18}$$

with $\bar{\alpha} = \bar{\rho}e^{j\bar{\Phi}}$ a complex gain, $\boldsymbol{\mu}(\bar{\boldsymbol{\eta}}) = (\mu_1(\bar{\boldsymbol{\eta}}), ..., \mu_K(\bar{\boldsymbol{\eta}}))^\top$ with $\mu_k(\bar{\boldsymbol{\eta}}) = a(kT_s - \bar{\tau})e^{-j2\pi f_c \bar{b}(kT_s - \bar{\tau})}$ for $k = 1, ..., K$, $\bar{\boldsymbol{\eta}} = (\bar{\tau}, \bar{b})^\top = ((\boldsymbol{P}_T - \boldsymbol{P}_R)/c, v/c)^\top$ and $\boldsymbol{n} = (n(T_s), ..., n(KT_s))^\top$ is the noise vector with pdf defined in (2). The parameter $\bar{b}$ is related to the Doppler frequency $F_d = \bar{b}f_c$. On the other hand, the misspecified signal model assumes white Gaussian noise as defined in Section 2.2. Following the results in Section 3 the pseudo-true parameters are $\boldsymbol{\varphi}_0^\top = [\sigma_0^2, \rho_0, \Phi_0, \tau_0, b_0] = [\sigma_0^2, \bar{\rho}, \bar{\Phi}, \bar{\tau}, \bar{b}]$.

5.1. MCRB for the contaminated model

Section 4 showed that the MCRB for the parameters of interest is equivalent to the Gaussian CRB up to a scaling factor $\bar{\sigma}_n^2(1 + (\bar{\kappa} - 1)\bar{\epsilon})$. Under these assumptions, an analytical expression can be derived thanks to recent work in [13] and the assumption of a bandlimited signal $a(t)$. More precisely, one obtains:

$$\mathbf{MCRB}^{-1}(\bar{\boldsymbol{\theta}}) = \frac{2F_s}{\bar{\sigma}_n^2(1 + (\bar{\kappa} - 1)\bar{\epsilon})}\mathrm{Re}\left\{\mathbf{QWQ}^H\right\}, \tag{19}$$

$$\mathbf{W} = \begin{bmatrix} w_1 & w_2^* & w_3^* \\ w_2 & W_{2,2} & w_4^* \\ w_3 & w_4 & W_{3,3} \end{bmatrix}, \quad \mathbf{Q} = \begin{bmatrix} e^{j\bar{\Phi}} & 0 & 0 \\ j\bar{\alpha} & 0 & 0 \\ j\bar{\alpha}\omega_c\bar{b} & 0 & -\bar{\alpha} \\ 0 & -j\bar{\alpha}\omega_c & 0 \end{bmatrix},$$

where the elements of $\mathbf{W}$ are functions of the baseband signal $\boldsymbol{a} = (a(T_s), ..., a(KT_s))^\top$:

$$w_1 = \frac{1}{F_s}\boldsymbol{a}^H\boldsymbol{a}, \quad w_2 = \frac{1}{F_s^2}\boldsymbol{a}^H\mathbf{D}\boldsymbol{a}, \quad w_3 = \boldsymbol{a}^H\boldsymbol{\Lambda}\boldsymbol{a}, \tag{20}$$

$$w_4 = \frac{1}{F_s}\boldsymbol{a}^H\mathbf{D}\boldsymbol{\Lambda}\boldsymbol{a}, \quad W_{2,2} = \frac{1}{F_s^3}\boldsymbol{a}^H\mathbf{D}^2\boldsymbol{a}, \quad W_{3,3} = F_s\boldsymbol{a}^H\mathbf{V}\boldsymbol{a},$$

with $K \times K$ matrices $\mathbf{D} = \mathrm{diag}(1, \cdots, K)$, $\boldsymbol{\Lambda}$, and $\mathbf{V}$ defined as:

$$(\boldsymbol{\Lambda})_{n,n'} = \left| \begin{array}{l} n' \neq n : \frac{(-1)^{|n-n'|}}{n-n'} \\ n' = n : 0 \end{array} \right., \tag{21a}$$

$$(\mathbf{V})_{n,n'} = \left| \begin{array}{l} n' \neq n : (-1)^{|n-n'|}\frac{2}{(n-n')^2} \\ n' = n : \frac{\pi^2}{3} \end{array} \right.. \tag{21b}$$

Using the above expressions, the MCRB can be computed for other signals under a chosen contamination model. This is useful for expanding this contribution to other practical applications of contamination in multi-sensor systems.

5.2. Validation

To validate the theoretical expressions shown in the previous sections, we adopt the MMLE formulation from [24] for joint time-delay and Doppler estimation:[1]

$$\hat{\boldsymbol{\eta}} = \arg\max_{\boldsymbol{\eta}} \left\|\boldsymbol{\Pi}_{\boldsymbol{\mu}(\boldsymbol{\eta})}\boldsymbol{x}\right\|^2, \tag{22}$$

We consider a scenario where a GPS L1 C/A signal [10] is received by a GNSS receiver that assumes an additive noise with a zero-mean Gaussian distribution. The true signal model is such that the noise has a complex contaminated distribution with parameters $(\bar{\kappa}, \bar{\epsilon}) = (20, 0.1)$ (scenario #1), $(\bar{\kappa}, \bar{\epsilon}) = (30, 0.05)$ (scenario #2), and $(\bar{\kappa}, \bar{\epsilon}) = (30, 0.3)$ (scenario #3). The signal-to-noise ratio at the output of the matched filter denoted as $\mathrm{SNR_{OUT}}$ is defined as:

$$\mathrm{SNR_{out}} = \frac{|\bar{\alpha}|^2\boldsymbol{a}^H\boldsymbol{a}}{\bar{\sigma}_n^2(1 + (\bar{\kappa} - 1)\bar{\epsilon})}. \tag{23}$$

Note that this expression reduces to the standard $\mathrm{SNR_{out}}$ when there is no contamination, i.e., when $\kappa = 1$ or $\epsilon = 0$ [13]. Fig.1 shows the root mean square error (RMSE) as a function of $\mathrm{SNR_{OUT}}$ for the MMLEs of (a) time-delay and (b) Doppler using a GNSS receiver with a sampling frequency of $F_s = 4$MHz and an integration time of 1 ms. The integration time refers to the duration of the GNSS signal that is observed. The RMSE is also shown as a function of the integration time for (c) time-delay and (d) Doppler when the SNR at the input of the receiver is fixed to $\mathrm{SNR_{IN}} = |\bar{\alpha}|^2/\bar{\sigma}_n^2 = -5$ dB. The results are averaged over 1000 Monte Carlo iterations. The RMSE for time-delay estimation approaches the predicted asymptotic performance, validating the proposed theoretical analysis. The RMSE of the Doppler MMLE also confirms the expected asymptotic performance in the same region as the time-delay MMLE. Since $\mathrm{SNR_{OUT}}$ is inversely proportional to $\bar{\sigma}_n^2(1 + (\bar{\kappa} - 1)\bar{\epsilon})$ while the MCRB is directly proportional to the same term, there is no change in the bound for different values of $\bar{\epsilon}$ and $\bar{\kappa}$ in (a) and (b). However, the estimation performance for a signal with a fixed SNR on arrival is shown to deteriorate with increased contamination parameters in (c) and (d). This means that in operating scenarios where the noise at the input can be sufficiently filtered, e.g., $\mathrm{SNR_{OUT}} = 15$ dB, the MCRB is achieved and is the same regardless of the values of $\bar{\epsilon}$ and $\bar{\kappa}$. Note that the higher the contamination level, the longer the signal must be to reach this minimum convergence threshold.

6. CONCLUSION

This work extended the misspecified Cramér-Rao Bound (MCRB) to contaminated Gaussian noise with outliers, modeled as a mixture of two Gaussian distributions with different variances. Analytical MCRB expressions and the asymptotic behavior of the corresponding MMLE were derived under a mismatched Gaussian assumption. An application to GPS time-delay and Doppler estimation showed that the MMLE error converges to the MCRB, which equals the classical Gaussian CRB with variance scaled by a function of the mixture probability and the ratio of inlier and outlier variances. This confirms the MCRB usefulness for evaluating estimators under realistic contaminated Gaussian noise assumptions. With the general form of the MCRB derived in this article, users interested in other multi-sensor applications that suffer from contaminated noise can easily determine their best possible estimation performance if they choose to keep a Gaussian assumption.

[1] If $S = \mathrm{span}(\boldsymbol{a})$ denotes the linear span of the set of the column vectors of a matrix $\boldsymbol{a}$, the orthogonal projector over S is $\boldsymbol{\Pi}_{\boldsymbol{a}} = \boldsymbol{a}\left(\boldsymbol{a}^H\boldsymbol{a}\right)\boldsymbol{a}^H$.

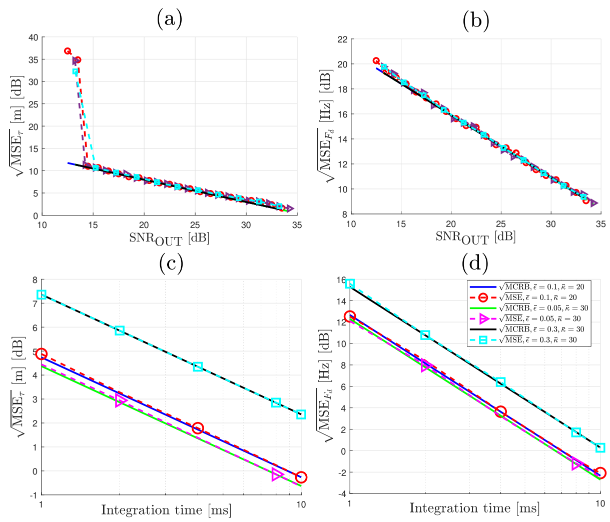

Fig. 1: MCRB and RMSE of the MMLEs of (a) time-delay and (b) Doppler for different SNR and fixed integration time of 1 ms, and for different integration times with a fixed $\text{SNR}_{\text{IN}} = -5$ dB (c) time-delay and (d) Doppler.

A. DERIVATION OF MATRICES $\mathbf{A}(\boldsymbol{\varphi}_0)$ AND $\mathbf{B}(\boldsymbol{\varphi}_0)$

A.1. Terms related to σ_n^2

Following the same step as in [16], we have:

$$\nabla_{\sigma_n^2}\nabla_{\sigma_n^2}^{\top} \ln f_{\boldsymbol{\varphi}}(\mathbf{x};\boldsymbol{\varphi}_0) = \frac{N}{\sigma_0^4} - \frac{2\text{tr}(\mathbf{n}\mathbf{n}^H)}{\sigma_0^6}, \tag{24}$$

where the equality between $\boldsymbol{\theta}_0$ and $\bar{\boldsymbol{\theta}}$ has been exploited. By computing the expectation w.r.t. the true data distribution $p_{\bar{\zeta}}$, computations similar to those used to obtain (9) lead to:

$$E_{p_{\bar{\zeta}}}\left[\nabla_{\sigma_n^2}\nabla_{\sigma_n^2}^{\top} \ln f_{\boldsymbol{\varphi}}(\mathbf{x};\boldsymbol{\varphi}_0)\right] = -\frac{N}{(\bar{\sigma}_n^2(1+(\bar{\kappa}-1)\bar{\epsilon}))^2}, \tag{25}$$

where the linearity of the expectation and trace operators has been used to invert their order. Similar operations yield:

$$E_{p_{\bar{\zeta}}}\left[\left(\nabla_{\sigma_n^2} \ln f_{\boldsymbol{\varphi}}(\mathbf{x};\boldsymbol{\varphi}_0)\right)^2\right] = E_{p_{\bar{\zeta}}}\left[\left(-\frac{N}{\sigma_0^2} + \frac{\mathbf{n}^H\mathbf{n}}{\sigma_0^4}\right)^2\right] \tag{26}$$
$$= \frac{N^2}{\sigma_0^4} - \frac{2N\text{tr}\,(\boldsymbol{\Sigma})}{\sigma_0^6} + \frac{E_{p_{\bar{\zeta}}}\left[(\mathbf{n}^H\mathbf{n})^2\right]}{\sigma_0^8} = \frac{E_{p_{\bar{\zeta}}}\left[(\mathbf{n}^H\mathbf{n})^2\right] - \sigma_0^4 N^2}{\sigma_0^8}.$$

Using the linearity of the expectation and the independence of the noise samples n_j and n_i for $j \neq i$, one obtains:

$$E_{p_{\bar{\zeta}}}\left[(\mathbf{n}^H\mathbf{n})^2\right] = E_{p_{\bar{\zeta}}}\left[\sum_{j=1}^{N}|n_j|^2\sum_{i=1}^{N}|n_i|^2\right] \tag{27}$$
$$= \sum_{i=1}^{N} E_{p_{\bar{\zeta}}}\left[|n_i|^4\right] + \sum_{i=1}^{N}\left(E_{p_{\bar{\zeta}}}\left[|n_i|^2\right]\sum_{j\neq i} E_{p_{\bar{\zeta}}}\left[|n_j|^2\right]\right)$$
$$= \sum_{i=1}^{N}(1-\bar{\varepsilon})E_{g_1}\left[|n_i|^4\right] + \bar{\varepsilon}E_{g_2}\left[|n_i|^4\right]$$
$$+ (1-\bar{\varepsilon})N(N-1)\bar{\sigma}_n^4 + \bar{\varepsilon}N(N-1)\bar{\kappa}^2\bar{\sigma}_n^4,$$

where $E_{g_1}[|n_i|^4]$ is the 4th order central moment of the Gaussian distribution with non-contaminated noise variance $\bar{\sigma}^2$, and $E_{g_2}[|n_i|^4]$ is the 4th order moment of the Gaussian distribution with contaminated variance $\bar{\kappa}\bar{\sigma}^2$. As a result, (26) simplifies to:

$$E_{p_{\bar{\zeta}}}\left[\left(\nabla_{\sigma_n^2} \ln f_{\boldsymbol{\varphi}}(\mathbf{x};\boldsymbol{\varphi}_0)\right)^2\right] \tag{28}$$
$$= \frac{-N^2\bar{\varepsilon}^2(\bar{\kappa}-1)^2 + \bar{\varepsilon}(\bar{\kappa}-1)\left((2N+1)\bar{\kappa}+2N-1\right)+2N}{(1+\bar{\varepsilon}(\bar{\kappa}-1))^4\bar{\sigma}_n^4}.$$

A.2. Terms related to $\boldsymbol{\theta}$

Following [16], one obtains:

$$\nabla_{\boldsymbol{\theta}} \ln f_{\boldsymbol{\varphi}}(\mathbf{x};\boldsymbol{\varphi}_0) = \frac{2}{\sigma_0^2}\sum_{k=1}^{K}\text{Re}\left\{n_k^*\nabla_{\boldsymbol{\theta}}\bar{f}_k\right\}, \tag{29}$$

where the notation $\nabla_{\boldsymbol{\theta}} f_k(\bar{\boldsymbol{\theta}}) = \nabla_{\boldsymbol{\theta}}\bar{f}_k$ has been used for brevity. Again, the equality between $\boldsymbol{\theta}_0$ and $\bar{\boldsymbol{\theta}}$ leads to:

$$\nabla_{\boldsymbol{\theta}}\nabla_{\boldsymbol{\theta}}^{\top} \ln f_{\boldsymbol{\varphi}}(\mathbf{x};\boldsymbol{\varphi}_0) = \frac{2}{\sigma_0^2}\sum_{k=1}^{K}\text{Re}\left\{\left(x_k - f_k(\bar{\boldsymbol{\theta}})\right)\left[\nabla_{\boldsymbol{\theta}}\nabla_{\boldsymbol{\theta}}^{\top}\bar{f}_k\right]^*\right\}$$
$$-\frac{2}{\sigma_0^2}\sum_{k=1}^{K}\text{Re}\left\{\nabla_{\boldsymbol{\theta}}\bar{f}_k\nabla_{\boldsymbol{\theta}}^{H}\bar{f}_k\right\}. \tag{30}$$

The matrix $\mathbf{A}(\boldsymbol{\theta}_0) \triangleq E_{p_{\bar{\zeta}}}\left[\nabla_{\boldsymbol{\theta}}\nabla_{\boldsymbol{\theta}}^{\top} \ln f_{\boldsymbol{\varphi}}(\mathbf{x};\boldsymbol{\varphi}_0)\right]$ is expressed as:

$$\mathbf{A}(\boldsymbol{\theta}_0) = -\frac{2}{\sigma_0^2}\sum_{k=1}^{K}\text{Re}\left\{\nabla_{\boldsymbol{\theta}}\bar{f}_k\nabla_{\boldsymbol{\theta}}^{H}\bar{f}_k\right\}, \tag{31}$$

since $E_{p_{\bar{\zeta}}}\left[x_k - f_k(\bar{\boldsymbol{\theta}})\right] = E_{p_{\bar{\zeta}}}\left[n_k\right] = 0, \forall k$. Note that (31) is related to the Fisher information matrix (FIM) of the well specified Gaussian case [11]. The values of the matrix $\mathbf{B}(\boldsymbol{\theta}_0)$ can be evaluated as follows:

$$\mathbf{B}(\boldsymbol{\theta}_0) \triangleq E_{p_{\bar{\zeta}}}\left[\nabla_{\boldsymbol{\theta}} \ln f_{\boldsymbol{\theta}}(\mathbf{x};\boldsymbol{\varphi}_0)\nabla_{\boldsymbol{\theta}}^{\top} \ln f_{\boldsymbol{\theta}}(\mathbf{x};\boldsymbol{\varphi}_0)\right]$$
$$= \frac{4}{\sigma_0^4}\sum_{k=1}^{K}\sum_{j=1}^{K}E_{p_{\bar{\zeta}}}\left[\text{Re}\left\{n_k^*\nabla_{\boldsymbol{\theta}}\bar{f}_k\right\}\text{Re}\left\{n_j^*\nabla_{\boldsymbol{\theta}}^{T}\bar{f}_j\right\}\right]$$
$$= \frac{2}{\sigma_0^4}\sum_{k=1}^{K}E_{p_{\bar{\zeta}}}\left[|n_k|^2\right]\text{Re}\left\{\nabla_{\boldsymbol{\theta}}\bar{f}_k\nabla_{\boldsymbol{\theta}}^{H}\bar{f}_k\right\}$$
$$= \frac{2}{\sigma_0^2}\sum_{k=1}^{K}\text{Re}\left\{\nabla_{\boldsymbol{\theta}}\bar{f}_k\nabla_{\boldsymbol{\theta}}^{H}\bar{f}_k\right\}, \tag{32}$$

with $E_{p_{\bar{\zeta}}}\left[|n_k|^2\right] = \sigma_0^2$ and using the fact that the true noise is i.i.d., i.e., $E_{p_{\bar{\zeta}}}\left[n_k^* n_j\right] = 0$ for $j \neq k$.

A.3. Cross-terms

Using the fact that the noise is zero-mean, a direct evaluation of the derivatives related to the cross terms yields:

$$E_{p_{\bar{\zeta}}}\left[\nabla_{\boldsymbol{\theta}}\nabla_{\sigma_n^2} \ln f_{\boldsymbol{\varphi}}(\mathbf{x};\boldsymbol{\varphi}_0)\right]^{\top} = \mathbf{0}_{1\times 4}, \tag{33}$$

$$E_{p_{\bar{\zeta}}}\left[\nabla_{\sigma_n^2}\nabla_{\boldsymbol{\theta}}^{\top} \ln f_{\boldsymbol{\varphi}}(\mathbf{x};\boldsymbol{\varphi}_0)\right] = \mathbf{0}_{1\times 4}, \tag{34}$$

$$E_{p_{\bar{\zeta}}}\left[\nabla_{\sigma_n^2} \ln f_{\boldsymbol{\varphi}}(\mathbf{x};\boldsymbol{\varphi}_0)\nabla_{\boldsymbol{\theta}}^{\top} \ln f_{\boldsymbol{\varphi}}(\mathbf{x};\boldsymbol{\varphi}_0)\right] = \mathbf{0}_{1\times 4}, \tag{35}$$

$$E_{p_{\bar{\zeta}}}\left[\nabla_{\sigma_n^2} \ln f_{\boldsymbol{\varphi}}(\mathbf{x};\boldsymbol{\varphi}_0)\nabla_{\boldsymbol{\theta}} \ln f_{\boldsymbol{\varphi}}(\mathbf{x};\boldsymbol{\varphi}_0)\right]^{T} = \mathbf{0}_{1\times 4}. \tag{36}$$

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