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RESEARCH PAPER

Non-analytic at the origin, behavioral models for active or passive non-linearity

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Most non-linear behavioral models of amplifiers are based on functions that are analytic at the origin and thus can be replaced by their Taylor series development around this point, e.g. polynomials of the input signal. Chebyshev Transforms can be used to compute the harmonic response of the model to a sine input signal. These responses are polynomials of the input signal amplitude. A second application of the Chebyshev transform to the first harmonic response or radio frequency (RF) characteristic will lend the carriers and intermodulation (IM) products for a two-carrier input signal, again polynomials. An important class of non-analytic non-linear behavior encountered in practice, such as hard limiters and detectors are either empirically treated or only approximated by an analytic function such as the hyperbolic tangent. This work proposes to generalize the polynomial non-linearity theory by adding non-analytic at the origin functions that, like polynomials, are invariant elements of the Chebyshev Transform. Devices modeled with these non-analytic at the origin functions exhibit intermodulation behavior significantly different from that of classical polynomial models, giving theoretical foundation to a number of important unexplained practical measurement observations.

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I. INTRODUCTION

Polynomial models are often used for non-linear memoryless behavioral models of amplifiers [1, 2]. When starting with an input-to-output transfer characteristic (e.g. the instantaneous voltage transfer characteristic of an amplifier), the Chebyshev Transform can be used to compute harmonic components of the output. The fundamental or first harmonic response is the AM/AM curve or radio frequency (RF) transfer characteristic. For polynomial models, this curve is also a polynomial because monomials are invariants of the Chebyshev transform [2]. Then the Chebyshev Transform may be used again to compute the levels of carriers and intermodulation (IM) products [3] for a two-carrier input signal. Their amplitudes are also given by polynomials of input signal amplitude. The polynomial model results can be applied to analytic functions models because they can be locally expanded as integer power series, their Taylor series developments at the origin (or Mac Lauren development).

This work shows that some non-analytic at the origin functions are invariant elements of the Chebyshev Transform in the same way the monomials are. This property simplifies the computation of the AM/AM curve, along with the output signal harmonics and two-carrier intermodulation products. It can be used also to easily invert Chebyshev transforms and to derive an instantaneous input-to-output model from the AM/AM curve. They can be used to model amplifiers in contradiction to propositions to reject models with discontinuity at the origin as not physical [4, 5].

The characteristics of these non-analytic at the origin models are substantially different from that of polynomial models. Their behavior may help to explain some puzzling practical measurement observations of intermodulation products in passive and active devices.

In many behavioral models, the use of functions of the modulus of the input (which is non-analytic at the origin) instead of the square of the modulus (which is analytic and can be derived from the classical theory) may result in an unexpected discontinuity. It may change the behavior of the model and it can be either physically acceptable or not.

II. OVERVIEW OF CLASSICAL THEORY

In the classical theory [1, 2], a real non-linear function f represents the instantaneous input-to-output transfer characteristic, e.g. a voltage transfer $V_{out} = f(V_{in})$.

When applying a sinusoidal input signal: $V_{in} = a\cos(\omega t + \varphi) = a\cos(\theta)$, the output signal is composed of harmonics of the input frequency, especially a DC component (or harmonic 0) and a fundamental component (or harmonic 1)

$$\begin{aligned} V_{out} &= f(V_{in}) = f[a\cos(\theta)] \\ &= \frac{1}{2} f_0(a) + \sum\nolimits_{m=1}^{\infty} f_m(a).\cos(m\theta). \end{aligned} \quad (1)$$

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Function $f_m(a)$ is the amplitude of harmonic m in the output signal. It is obtained as the order m Chebyshev transform of $f(u) = f[a\cos(\theta)]$

$$f_m(a) = \frac{1}{\pi}\int_{-\pi}^{+\pi} f[a\cos(\theta)]\cos(m\theta)d\theta. \quad (2)$$

The output amplitude at fundamental frequency, $f_1(a)$, always an odd function of a, is called the AM/AM curve. The fundamental gain curve, or describing function (DF), $g(a) = f_1(a)/a$, is always an even function of a. The Chebyshev transform is a special case of the discrete Fourier transform: it is the discrete cosine transform of the output signal when the input signal is a cosine and so is periodic and even. Function f may be non-analytic. Mathematically, it can be any distribution for which the Chebyshev transform can be computed. However, the classical theory generally restricts itself to analytic functions and frequently to polynomials.

For a function f that is an integer power of the variable u: $f(u) = u^n$, the transform is [2]

$$\begin{aligned} f_m(a) &= \frac{1}{\pi}\int_{-\pi}^{+\pi} [a\cos(\theta)]^n \cos(m\theta)d\theta \\ &= 2\left(\frac{a}{2}\right)^n \frac{n!}{\left((n+m)/2\right)!\,\left((n-m)/2\right)!} \end{aligned}. \quad (3)$$

Note that equation (3) is valid only for integer m of same parity as n, with $0 \le m \le n$. The result of the integral is null in other cases. Power functions (or monomials) are therefore invariants of the Chebyshev transform: the transform of a power function is a power function with the same exponent. As a consequence, if the function f is a polynomial, then all its Chebyshev transforms are polynomials. Even (respectively odd) harmonics are given by the terms of even (respectively odd) powers of input signal. Consequently, the fundamental output amplitude, $f_1(a)$, is given by a polynomial with only odd degree terms. In case of two or more carriers input, the intermodulation products around the fundamental carriers will have odd orders only. Finally, the gain curve, $g(a) = f_1(a)/a$, is an even polynomial of input amplitude (with only even degree terms) or an arbitrary polynomial of input signal power (when power is proportional to the square of the input signal: voltage, current or wave).

The interest of these invariants, as shown by Blachman [3], lies in the possibility to iterate the Chebyshev transform (once or many times). This makes computation of intermodulation products easy for two-carrier and multicarrier signals. In addition, the Chebyshev transform can be inverted easily when approximated with a polynomial.

This presentation is limited to the AM/AM curve and real polynomials but AM/PM can easily be taken into account with complex coefficients polynomials as shown in [6]. All of the results are formally the same. The AM/PM phase curve can then be modeled as an even function of input amplitude in the same way as the gain. The AM/AM and AM/PM curves are identical to the amplitude and phase of the sinusoidal-input DF in control theory [7]. The function depends only on signal amplitude and not on signal frequency when the non-linearity is memoryless (called static in control theory).

A drawback of the polynomial models is their rapidly divergent behavior as the input amplitude goes beyond the maximum input value that has been used while characterizing the amplifier. It can be a problem for telecom simulation where Gaussian noise or multicarrier input signals with high dynamic range (e.g. orthogonal frequency division multiplex (OFDM) signals) have peaks much higher than their average value [8]. To overcome this drawback, without introducing a limiter, a number of analytic functions have been proposed as models such as trigonometric (or Fourier–Bessel) series decompositions, rational functions or the hyperbolic tangent. Saleh proposed a rational function model of travelling wave tubes [9] that is widely used. Cann [10] proposed a model for solid-state power amplifiers (SSPA) in addition to limiters and hyperbolic tangents. Rapp [8] used the Saleh model and also proposed a modified model for the SSPA. A more complete discussion of the SSPA models will be found in Schreurs *et al.* [11].

To conclude, the main inferences of the classical analytical function theory are the following:

1. Orders of intermodulation products around the fundamental signal output (i.e. the odd harmonic of order 1) are odd integers: 3, 5, 7, etc.
2. A polynomial function will give only products of order lower or equal to its degree.
3. In small signal operation, the level of an intermodulation product will follow a slope, in dB/dB, with respect to the input level, equal to an integer number.

   This slope equals the lowest degree in the polynomial that is higher or equal to the order of the intermodulation product.

   Generally, this is understood as: third-order intermodulation products have a slope of 3 dB/dB, fifth-order 5 dB/dB, and so on. However, if the third degree term in the polynomial is null, whereas the fifth degree term is not, then the third-order product would have a slope of 5 dB/dB.

III. SIMPLE NON-ANALYTIC MODELS

A number of simple non-analytic models are commonly used to represent real world device operations: threshold, limiters, class AB, B, and C amplifiers or detectors as illustrated Fig. 1.

Owing to the lack of a general theory for non-analytic functions, the harmonic and intermodulation content at the output of such devices is in general computed directly for

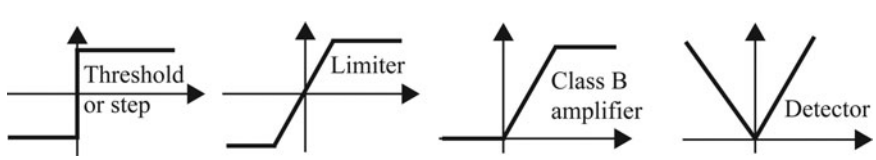

Fig. 1. Some ideal non-analytic models.

each case, using for example Fourier decomposition of square waves for the threshold function. No general rules have been drawn for the behavior of harmonics and intermodulation products.

Other work exists in this domain, particularly in control theory [7, 12], two-carrier or multicarrier input signal is taken into account with multiple-input DFs. They are used to linearize the system and express amplitudes of output signals at frequencies of input signal. Noise and intermodulation products at frequencies different from those of the input signal are considered generally as errors in linearization.

Piecewise linear, piecewise polynomials or cubic splines are used to model transistors measured curves such as drain voltage versus drain current curve as a function of gate voltage for simulation purposes but not to obtain closed form results. Examples of DFs for such non-analytic functions are given in [7, 12–14].

Blachman [2] gives the Chebyshev transform of some non-analytic functions, for example, the product of a power function and Heaviside function H given by

$$f(u) = H(u) \cdot u^n \quad \Leftrightarrow \quad f(u) = u^n, \quad \text{for} \quad u > 0 \quad \text{and} \quad f(u) = 0, \quad \text{for} \quad u \leq 0. \tag{4}$$

Its transform is:

$$f_m(a) = \left(\frac{a}{2}\right)^n \frac{n!}{\Gamma\big((n+m/2)+1\big)\,\Gamma\big((n-m/2)+1\big)}. \tag{5}$$

Note that the factorial functions in the denominator of (3) have been replaced by the Gamma function in (5) as m and n may not have the same parity and the arguments may become half-integer. Since the Heaviside function is neither odd nor even, it generates even and odd harmonics or products. In particular, the output includes all harmonic orders of parity opposed to that of the degree n, even for a constant ($n = 0$) or a linear polynomial ($n = 1$) and harmonics of the same parity as degree n up to n: $m = n$, $n - 2$, $n - 4$, ... with $m \geq 0$.

IV. NEW INVARIANTS OF THE CHEBYSHEV TRANSFORM

In this paper, the presentation will be restricted to functions that are not smooth (not C^∞ and so not analytic) only at the origin. By definition of smoothness (C^∞), either the function or one of its derivatives is not continuous at the origin and higher-order derivatives are infinite or undefined. Such a function cannot be replaced by a polynomial or an integer series around 0, even in a small range. In Fig. 1, only the threshold function and the linear detector are in this case. A class B amplifier without limitation (half-wave rectifier) would also be in that case.

A non-linear function based on the terms given in (4) could be used to model a non-linear class B amplifier in the small signal range. A push–pull amplifier would be obtained by complementing the positive part of the function with its opposite for negative inputs, thus obtaining an odd function. Even degree parts of this non-linear function do not disappear as they would if it was forced to be analytical and they are the cause of non-analyticity.

This is obtained by replacing the Heaviside function in equation (4) by the *sign* function:

$$F(u) = \text{sign}(u) \cdot u^n = \{H(u) - H(-u)\} \cdot u^n. \tag{6}$$

The sign function is odd and the parity of function f above is opposed to that of exponent n.

Its transform is:

$$f_m(a) = 2.\,\text{sign}(a).\left(\frac{a}{2}\right)^n \frac{n!}{\Gamma\big((n+m/2)+1\big)\,\Gamma\big((n-m/2)+1\big)}. \tag{7}$$

Equation (7) is valid for all values of m with parity opposed to that of n. In consequence, there are no harmonics or products of order having the same parity as n. More importantly, equation (7) shows that powers of the input multiplied by sign function are invariants of the Chebyshev transform like classical monomials. The polynomial model is generalized as

$$f(u) = P(u) + \text{sign}(u).Q(u) \tag{8}$$

$P(u)$ and $Q(u)$ are arbitrary polynomials of input signal u. Both may contain odd terms and even terms. Here, $P(u)$ is the polynomial that would be used in the classical theory.

It is, however, more appropriate to depart from the classical theory and explicitly separate the function in odd and even parts, using different polynomials

$$f(u) = P(|u|) + \text{sign}(u).Q(|u|). \tag{9}$$

The model of the odd part of the device characteristic: $\text{sign}(u).Q(|u|)$ is generally the only one that gives output in the useful RF bandwidth. The even part $P(|u|)$ should be considered when interested by the DC component (harmonic 0) and second harmonic output or when the bandwidth is large compared to centre frequency. Invariants of the Chebyshev transform are

$$\text{Even invariants, for any integer } n\text{: } f(u) = |u|^n, \tag{10}$$

$$\text{Odd invariants, for any integer } n\text{: } f(u) = \text{sign}(u).|u|^n. \tag{11}$$

The Chebyshev transforms of these invariants are readily given by

Even transforms:

$$f_m(a) = 2.\left(\frac{|a|}{2}\right)^n \frac{n!}{\Gamma\big((n+m/2)+1\big)\,\Gamma\big((n-m/2)+1\big)} \quad \text{for even integer } m, \tag{12}$$

Odd transforms:

$$f_m(a) = 2.\text{sign}(a).\left(\frac{|a|}{2}\right)^n \frac{n!}{\Gamma((n+m/2)+1)\,\Gamma((n-m/2)+1)}$$

for odd integer m. (13)

Observe that the parity property (even or odd) is maintained throughout the Chebyshev transform. This implies that, as in the classical theory, orders of intermodulation products around the fundamental carrier signal output will be odd integers (3, 5, 7...) and will result only from the odd part of function $f(u)$.

There are two important differences with the classical analytic model theory:

- An odd invariant of any given degree (except an odd integer), will give harmonics and intermodulation products of all odd orders, instead of only orders lower than or equal to its degree (and similarly for even invariants).
- Under small signal conditions, unlike in the classical analytic function theory, the output amplitude of a harmonic or product is no longer a monomial of the input amplitude with an integer exponent equal at least to its order m and with the same parity. The exponent (and so the slope in dB/dB) is equal to the degree n and not to the order m. In small signal conditions, a third order intermodulation product level does not always vary as a function of input level with a slope of 3 dB/dB. The slope could be any even integer (0, 2, 4...) as well as classically an odd integer at least equal to its order (3, 5...).

The classical constraints on the slopes of small signal intermodulation products (3 dB/dB, 5, 7, etc.) are not physical but an artifact due to the use of analytic invariants (the monomials), polynomials and analytic functions to describe or approximate the non-linearity.

Since the fundamental carrier output will result from the odd part of the characteristic, sign(u).Q(|u|), the AM/AM curve will be under the form: $f_1(a) = \text{sign}(a).R(|a|)$ with the polynomial R depending on the polynomial Q. The fundamental gain (or DF), an even function of a will be under the form

$$g(a) = f_1(a)/a = \text{sign}(a).R(|a|)/a = R(|a|)/|a|. \quad (14)$$

As in the classical theory, one can use an even function to model the AM/PM curve and model its effect on the complex envelope of the signal: $\tilde{u} = a \cdot e^{j\theta}$. Function sign($\tilde{u}$) is not defined on complex numbers and it must be replaced by $\tilde{u}/|\tilde{u}| = e^{j\theta}$.

Now, in light of the results above, a more general family of real and complex non-analytic invariants can be proposed, where the exponents can be fractional or real numbers p

$$\text{Even invariants, for any real } p\text{: } f(u) = |u|^p \text{ or } f(\tilde{u}) = |\tilde{u}|^p \quad (15)$$

$$\text{Odd invariants, for any real } p\text{: } f(u) = \text{sign}(u).|u|^p \text{ or } f(\tilde{u}) = \tilde{u}.|\tilde{u}|^{p-1} \quad (16)$$

The Chebyshev transforms of the above invariants are formally given by the same equations as (12) and (13) with integer n replaced by real p and factorial $n!$ replaced by Gamma function $\Gamma\ (p+1)$. These non-analytic functions are also invariants of the Chebyshev transform and obey the same laws as the monomials provided that the real exponent p is higher than -1 to guarantee mathematical convergence of the series of transforms [2].

V. PHYSICAL CONSTRAINTS ON NON-ANALYTICITY

For these invariants to be physical models of transfer functions, they must not create energy. This introduces additional constraints on the exponent

1. The exponent must be positive or null to guarantee a finite output for null input for any physical device (an infinite output without input would not be acceptable).
2. The exponent must be at least equal to 1 to guarantee a finite gain, and null output for null input, for a passive device (not necessary for an active device with a power supply).

For a positive real exponent p, the invariant is continuous but, except for integer p of the same parity as the invariant, its derivatives of order higher than p are not continuous at the origin and derivatives of order higher than $p+1$ are indefinite. They are not smooth (C^∞). A model based on these invariants will not be smooth, and so will not be analytic, at the origin. This is acceptable because the energy in the output signal is bounded when the input energy is bounded: bounded-in, bounded-out condition [7].

By using invariants (15) and (16) to model a device characteristic, one can explain small signal harmonics and IM products slopes of any positive or zero real value in dB/dB. For example, the output of a hard limiter or trigger, an active device with odd non-linearity and $p = 0$, consists of all odd harmonics and intermodulation products, each at a finite and constant level: all slopes are 0 dB/dB. The gain can be expressed by replacing the polynomial R in (14) by a sum or series of terms with real exponents p. This will give the designer much more flexibility for modeling practical measurement data, first by using odd and even degrees in polynomials of the modulus of the input signal, second by complementing the function with terms having real valued exponents.

The fact that a non-analytic at the origin model is better at modeling the behavior of a device is not a demonstration that the physical device itself has a discontinuity of some of its derivatives. Much more complicated analytic (or at least smooth) functions could be used to model the same behavior down to the level of thermal or quantum noise, where measurements would no longer be possible.

A simple example would be to replace the sign(u) function by the hyperbolic tangent: tanh(ku), with a large enough multiplicative coefficient k. The sign function is the mathematical limit of the hyperbolic tangent when k tends to infinity and it gives qualitative and easy to understand results that can be compared to measurements. The hyperbolic tangent model is continuous and infinitely derivable but the results are more complicated and would not permit easy iteration and inversion of the Chebyshev transform.

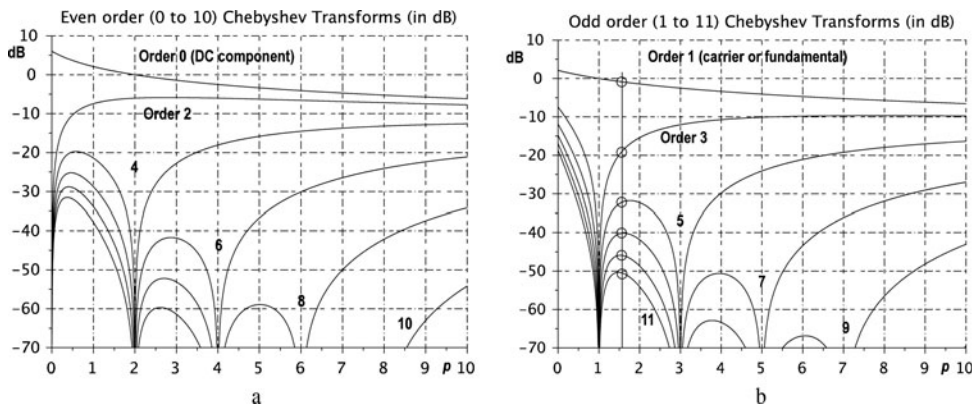

Fig. 2. Harmonics (or products) levels as a function of real exponent p in power function: (a) even order (0 to 10) transforms of even functions of type $f(u) = |u|^p$ (b) odd order (1 to 11) transforms of odd functions of type $f(u) = sign(u).|u|^p$

On the other hand, the simplicity and efficiency of a model based on non-analytic at the origin functions may be a good reason to examine some physical processes and try to find demonstrations, based on physics, of the reality or not of the discontinuity at the origin.

VI. BEHAVIOR OF CHEBYSHEV TRANSFORM INVARIANTS

The output level of harmonics for a sinusoidal input (or equivalently IM products for two-carrier input) as a function of real exponent p are shown in Fig. 2 for the even parity and odd parity invariants (15) and (16).

In the case of integer exponents, harmonics or products disappear, in agreement with the classical theory, when they have the same parity as the exponent and strictly higher order. The curves are continuous functions of p, they only seem to be discontinuous on a graph in dB.

VII. ANALYSIS OF SOME MEASUREMENT RESULTS

A) Passive devices: antennas and filters

Many authors [15–18] have reported even integer and/or non-integer slopes for odd order intermodulation products in passive devices such as antennas, filters and lines. These slopes were either explicitly presented as unexplained or supposed to be measurement errors.

The authors in [16] measured passive intermodulation products of orders 3, 5, 7, and 9 in a micro strip line with 14 dB input level range around 900 MHz. They report slopes between 1.6 and 2.5 dB/dB for third-order to ninth-order IM products. Ratios of successive products are: $I3/I5 = 14$ dB, $I5/I7 = 15$ dB and $I7/I9 = 10$ dB. They also report slopes between 2 and 2.9 dB/dB on two different coplanar waveguide lines. They did not give an explanation for these unconventional values. Measurement data from [16] is presented in Fig. 3.

A simple non-analytic model of the micro strip line, using only one term with an exponent of 1.6, would give a slope of 1.6 dB/dB for all intermodulation products and ratios between successive products of: 13.5, 8 and 6 dB; see Fig. 2(b) at abscissa $p = 1.6$.

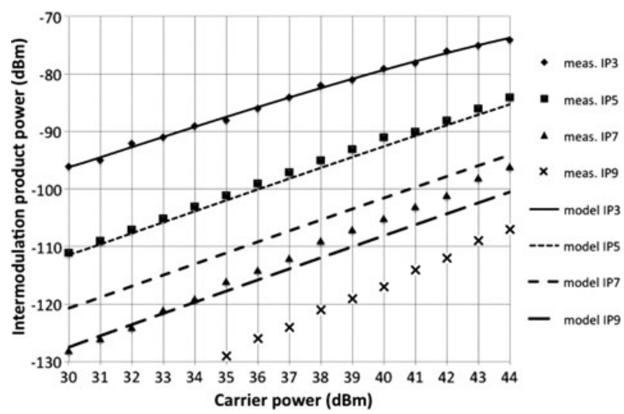

Fig. 3. Comparison of measured levels (dots, data from [16]) and modeled levels (lines, this work) of intermodulation products of orders 3 to 9.

A second model with two terms (exponents 2 and 2.5) has been adjusted only to third-order IM product measurements from [16]. It shows even better agreement (see Fig. 3), resulting in quadratic average errors of 0.4 dB for third IM level (at -90 dBm) but also around 1 dB for fifth IM level (at -100 dBm) in a 14 dB input power range. Errors of 5 dB for seventh IM level and 9 dB for ninth IM level are pessimistic but acceptable at the level of -120 dBm.

The model can then be used to simulate efficiently and precisely the device behavior for other input signals, CW or modulated, and to compute noise power ratio for a multicarrier signal.

The authors of [17, 18] report on measurements of third order intermodulation products on base stations towers and equipment giving non-integer slopes of approximately 2 dB/dB.

A polynomial model of order 49 has been used in [17] with unsatisfying results.

B) Active devices: mixers and amplifiers

In [19], the authors have measured the non-linear voltage to current response of one NMOS cold FET used in a double balanced mixer. They computed three models of this current: a third degree, a tenth degree and a piecewise third degree polynomial approximation. This last approximation gave a much better fit than the first two, with an error 40 dB lower. However, the authors finally rejected this approximation because it gave a 2 dB/dB small signal slope for the third-order intermodulation product in BSIM3 simulation, which they did not find in their measurements. The second-degree term, present in their single transistor measurement, could have been partially masked because of the double balanced mixer structure or the higher LO power operation. Lower power measurement could better identify if a residual 2 dB/dB small signal slope exists.

In contradiction to [19], the authors in [13, 14] have used a model equivalent to BSIM3 that includes a second-degree term in the drain current to drain voltage characteristic of the transistor. Using harmonic balance (HB) software, they simulated amplifiers and obtained small signal intermodulation products with slopes varying from 2 to 3 dB/dB, for different classes of amplifiers: A, AB, or B. This was confirmed by the measurements in [13] and explained by the second-degree term in the model in [14]. This is in line with the theory using even and odd powers of the modulus. However, with such a model, the slope should decrease to 2 dB/dB for very low signal power. If the measurements confirm that the small signal slope value is constant and different from 2 (real value between 2 and 3), then a model with a real exponent would give a better approximation of the device.

VIII. APPLICATION TO SSPA MODELS

A) Volterra models

Modulus of the complex envelope of input is used in Volterra models [20–22]. In some cases, this is simply an empirical way to express that the non-linear gain depends on the amplitude or power of the input signal but not on the input signal phase. However, the modulus is continuous but its derivative is discontinuous at origin. This means that the underlying theory must be based on non-analytic at the origin functions, instead of polynomials, if odd powers of the modulus are used in the gain. In that case, the results obtained from non-linear polynomials and analytic functions theory cannot be applied without caution. The authors in [4] explain that only even powers of the modulus should be used to avoid discontinuity at the origin and stay in the classical theory. This is true but the classical theory is too restrictive.

The authors in [21, 22] introduce and validate the polar Volterra model. The nth sample of the complex envelope of the output signal is given in [21] by

$$\tilde{y}(n) = \tilde{h}_{0,0} + \sum_{p_1=0}^{P_1} \sum_{p_2=0}^{P_2} \sum_{m_1=0}^{M} \cdots \sum_{m_{p1}=m_{p1-1}}^{M} \sum_{l_1=0}^{L} \cdots \sum_{l_{p2}=l_{p2-1}}^{L} \sum_{l_{p2+1}=0}^{L} \cdots \sum_{l_{2p2-1}=l_{2p2-2}}^{L} \times\Big[\tilde{h}_{p_1, 2p_2-1}(m_1, \ldots, m_{p_1}, l_1, \ldots, l_{2p_2-1}) \cdot a(n-m_1)\cdots a(n-m_{p_1})\cdots e^{j\phi(n-l_1)}\cdots e^{j\phi(n-l_{p_2}}). \cdot e^{-j\phi(n-l_{p_2+1})}\cdots e^{-j\phi(n-l_{2p_2-1})}\Big], \quad (17)$$

where $\tilde{x}(n) = a(n) \cdot e^{j\phi(n)}$ is the nth sample of the complex envelope of the input signal.

Discarding the memory in the model, all phase terms in (17) collapse except one. If combined with one of the moduli, it gives back the complex envelope of the input signal. Then, the output envelope is expressed as a product of fundamental gain and input signal envelope

$$\tilde{y}(n) = \tilde{h}_{0,0} + \tilde{x}(n) \cdot \sum_{p_1=0}^{P_1} \Big[\tilde{h}_{p_1} \cdot a^{p_1-1}(n)\Big]. \quad (18)$$

The non-linear gain in (18) is a function of the modulus of the input signal. All integer degrees can be present with a minimum of -1. This is acceptable for an active non-linearity. It should be restricted to $p_1 \geq 1$ for a passive non-linearity. The non-linearity is non-analytical at the origin when odd degrees are used in the gain (even values of p_1).

Equations (17) and (18) should not be restricted to integer powers of the modulus. Real exponent values should be taken into account (at least for the AM/AM curve) when necessary to model measured even-integer or non-integer slopes of harmonics or IM products.

B) Large signal scattering parameter models

The modulus of the input signal complex envelope is also used in the definition of large signal scattering parameters in [23]. The large signal scattering parameter $X_{21,11}$ is the fundamental gain or DF between components at fundamental frequency of the input and output waves. It can be obtained in a polyharmonic measurement in a large signal network analyser (LSNA). Calibration equations for the LSNA also use the modulus of the input wave [24]. All of these equations and definitions must obey the constraints described in Section V and should contain only even powers of the modulus if there is no discontinuity of any derivatives at the origin. On the contrary, odd integer and real exponents must be used to model non-linear S-parameters of devices with discontinuity at the origin and with harmonics or IM products small signal behavior different from the one derived from the classical theory.

C) Modified Saleh models

Modified Saleh models given in [8, 11] have been proposed by the authors (and generally been used) with integer values of parameters that make them analytic or their small signal behavior was not investigated. The modified Saleh model proposed by Cann [10] was criticized in [5] as not being analytical at the origin and so not being physical. The author remarked that the behavior of simulated two-carrier intermodulation products did not obey the classical theory and “had never been measured”. This may depend on the parameter and the exact algorithm used for simulation. Cann proposed a revised model [25] that is analytical.

Clearly, as shown in this work, many authors have measured even integer and non-integer dB/dB slopes for odd IM products in small signal conditions, on active and on passive devices, and this cannot be explained with an analytic model. Hence, a model cannot be rejected only because of this behavior. The correct test is for the model to behave (or not) as the physical device does in the input power range of interest.

For the measurement examples given in this work, a non-analytic at the origin model has been found to be the simplest one to obtain this behavior together with a better approximation of the transfer function in a greater range than polynomial models.

Clearly also, a non-analytic at the origin model should be used only if it is simpler than the polynomial model and if it better fits the transfer curve and the IM products' small signal slopes measured on the physical device.

IX. CONCLUSION

This work shows that:

1. In small signal conditions, the level of a harmonic or product at the output of a non-linear function may vary with a dB/dB slope versus the input level different from its order. Even integer slopes and real valued slopes have been measured and reported for odd IM products by many authors. They are explained in this work by introducing new non-analytic invariants of the Chebyshev transform.
2. A term of any real power (except an even integer) of the input modulus will give harmonics and intermodulation products of all even orders
3. A term of any real power (except an odd integer) of the input modulus, multiplied by the *sign* function, will give harmonics and intermodulation products of all odd orders.

No physical or mathematical reasons permit to restrict non-linear functions used to model passive or active functions to polynomials or analytic functions. In fact, some measured behavior cannot be explained with polynomials or analytic functions only.

A real exponent power series should be used instead of polynomials where it permits to better approximate measurement data and model non-linear passive or active functions.

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