A simple and robust K-factor computation method for GNSS integrity needs

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***Abstract*—The aviation Minimum Operational Performance Standard defines the SBAS protection levels as the product of the estimated standard deviation of the positioning error and a scaling factor called K-factor. The K-factor depends on the time window of interest and on the correlation between errors in the time window. The K-factors defined in aviation are difficult to generalize to other specifications in other domains, such as rail and maritime applications.**

This article presents a simple formula to calculate the K-factor for any value of integrity risk and time interval. The resulting K-factor is shown to be mathematically rigorous under the hypothesis of Gaussian error distribution but without any assumption on the correlation structure of the successive position estimates. The Gaussian assumption can be relaxed and replaced by overbounding with a Gaussian distribution with a very good approximation. This formula can be used in any GNSS application where integrity is needed.

***Index Terms*—K-factor, protection level, integrity risk, GNSS augmentation, SBAS**

I. Introduction

In GNSS applications, integrity is defined as a measure of the level of trust a user can place in a position estimate. The concept of integrity was originally developed in the field of civil aviation for Safety-of-Life applications, such as commercial aircraft landing. Nowadays, its generalization beyond aviation is an area of intensive research since it is the prerequisite to the operational deployment of the emerging Safety-of-Life applications such as autonomous transportation (e.g. train, car, taxi-drone).

Integrity is based on the definition of four specific features: the alarm limit (AL), the time to alarm (TTA), the integrity risk (IR) and the protection level (PL). The alarm limit is the position error (PE) threshold not to be exceeded without triggering an alert. The TTA is the maximum allowed time that elapses between when the navigation system is out of tolerance and when the equipment triggers the alert. The position estimate is a random variable, distributed according to a specific cumulative distribution function. Hence we can compute the probability P that it falls outside of the considered confidence interval, which is called protection level (PL).

Integrity is compromised when anomalies occur that can cause unpredictable PE beyond the operational PL for more than TTA consecutive seconds over a time interval T. The integrity risk (I_R) is then defined as the probability that the true position lies outside of the PL at any time within a time interval T, and for TTA consecutive seconds. As an example, for aviation precision approach operations these parameters are $I_R = 10^{-7}$ for a duration $T = 150$ s, with a TTA of 6 s. The specifications of the Minimum Operational Performance Standard (MOPS, [1]) define the PL as the product of the variance standard deviation of the positioning error and a scaling factor called K-factor. These K-factors can also be understood as quantiles of the normalized residual errors distribution, sometimes known as safety index (SFI). The tuning of the K-factor is critical for user performances: a large protection level ensures a larger margin on IR but degrades the availability and continuity of the Satellite Based Augmentation System (SBAS) for instance.

The statistical study of the behaviour of the positioning errors is complicated by the fact that successive position estimates are not independent variables. This is because the SBAS system provides corrections for satellite orbit determination, tropospheric and ionospheric delay, which are physical phenomenon with longer intrinsic time-scales than the frequency of GNSS measurements (typically one estimate per second). Furthermore, errors from the user's environment such as multipath also affect the correlation of the position errors in a complicated way.

Taking this time correlation into account, the MOPS sets K-factor values for every approach covered by SBAS (see [1] and [2]). For example, $K = 5.33$ for vertical protection level in precision approach. This value comes from assuming that all errors are perfectly correlated in a time window of 360 seconds and corresponds to the 10^{-7} quantile of a 1D Gaussian distribution. However, the MOPS recognizes that this modelling of the errors' time correlations is somewhat arbitrary. This makes it difficult to generalize to other specifications outside of aviation, like maritime and rail applications.

To tackle this problem, Antic, Maliet & Trilles [3] introduced the Gauss Markov K-factor (GMK), which uses PSD-overbounding techniques to find a rigorous upper bound for the K-factor for any specifications. It improves upon the MOPS standard with a mathematically justified method and with a finer modelling of the time correlation. We show that in practice, the GMK method has two main limitations: first, it often leads to over-conservative K-factors due to the PSD-overbounding step, and second, it is difficult to generalize to non-Gaussian tails.

In this paper, we suggest a simple method to determine the K-factor for any specification of integrity risk and time window. We provide a ready to use analytical formula for the K-factor as function of I_R and the time window T that can be applied to any field. Then we prove mathematically that, under the assumption of Gaussian distribution, the K-factor in the independent case is a rigorous upper bound of the optimal K-factor for any correlation structure of the time series. Furthermore, we show that this result is still valid if we relax the Gaussian distribution assumption and replace it by a Gaussian CDF-overbounding assumption, which is commonly used in the navigation community.

II. Definitions and assumptions

We consider a series of GNSS position estimation done periodically (typically one estimate per second). The collection of N estimates in the time window of interest will be denoted $Z = (z_n)_{1\leq n\leq N}$. Z is a vector of N random variables and we will assume that it follows a Gaussian distribution, centred on the true position of the receiver $Z^* = (z_n^*)_{1\leq n\leq N}$. The precise assumptions on the probability distribution of Z are detailed in the next subsection II-A.

The concept of integrity is specified by the integrity risk I_R, the time window T and the time to alert (TTA). For example, for aviation precision approach, $I_R = 10^{-7}$, $T = 150$ s and $\mathrm{TTA} = 6$ s. The goal of the integrity monitoring is to build protection volumes V_n around each estimates z_n such that the true position z_n^* never exits V_n for more than TTA consecutive seconds [4]. The probability of such an event during the time window T has to be maintained below I_R.

In order to treat this problem mathematically, we will use an approximate but conservative definition of integrity. We will say that integrity is maintained if the true position always lies in the volume during the entire time window with very high probability:

$$\mathbb{P}\left(\forall n \in [\![1, N]\!],\ z_n^* \in V_n\right) \geq 1 - I_R \tag{1}$$

When the period between each z_n is smaller than TTA, it is straightforward to see that the condition (1) implies the condition of integrity given above (if every real position is in the corresponding protection volume, there is no exit from the protection volumes for more than TTA consecutive seconds).

From this remark, we see that enforcing condition (1) on a subset of position estimates regularly separated in time by at most TTA is enough to guarantee integrity. So in general, the number of estimates N one needs to consider in equation (1) is given by the ceiling of the ratio T/TTA.

A. Positioning errors distribution

In all what follows, we are making the following assumptions about the probability distribution of the positioning estimates Z.

- The dimension of the positioning measurement is denoted d: for example in case of vertical positioning (for aircraft), we have $d = 1$, in maritime positioning $d = 2$. In what follows, we will consider the case $d = 1$ for simplicity (so that $z_n \in \mathbb{R}$), however the generalization of all our results to $d \geq 2$ is straightforward and is detailed in Appendix A.
- The position estimators are unbiased, so that the mean of the positioning errors is zero. In other words, $E(z_n) = z_n^*$.
- The distribution of the errors follow a multidimensional Gaussian law. This is a natural but very strong assumption that allows us to benefit from the numerous properties of Gaussian distributions. This means that the probability distribution is completely determined by the correlations between each measurements summarized in the square matrix $\Sigma_{ij} = \mathrm{cov}(z_i, z_j)$. This assumption is necessary for the GMK method presented in section III-B. It is also important for the method of section V but it can be relaxed to a more realistic assumption.
- The standard deviation at each position estimation is accurately measured by the system and accessible to the user. This is also a strong assumption because it requires the system to have a full understanding of its error sources. We will also relax this assumption in section V.

B. Definition of the K-factor and building of the protection volume

The protection volumes V_n are built through the notion of K-factor (see [1] appendix J). At each position estimate, we assume that the system measures accurately the error of z_n (in other words, the system measures the covariance $\sigma_n^2 = \Sigma_{nn} = \mathrm{Var}(z_n)$. From this, we define the protection volume as an interval centred on the estimator output with radius equal to K times the standard deviation:

$$V_n = [z_n - K\sigma_n,\ z_n + K\sigma_n] \tag{2}$$

The goal of integrity studies is to determine a K-factor such that the inequality (1) holds, while keeping K small enough to have reasonable protection volumes (2).

C. Infinity norm of a normalized Gaussian vector

The way the K-factor is built encourages us to work with the vector of reduced errors $X = (x_n)_{1\leq n\leq N}$ (sometimes known as safety index or SFI) defined by

$$x_n = \frac{z_n - z_n^*}{\sigma_n}$$

Then X follows a Gaussian distribution with zero mean and covariance matrix C which is linked to the estimates covariance matrix by the formula:

$$C_{ij} = \mathrm{cov}(x_i, x_j) = \frac{\Sigma_{ij}}{\sigma_i \sigma_j}$$

In particular all diagonal elements C_{ii} are equal to 1, meaning that each reduced error x_i follows a centred normalized normal distribution when marginalized over all other measurements.

The definition of the protection volume (2) becomes simpler with the reduced errors:

$$\left(\forall n \in [\![1, N]\!],\ z_n^* \in [z_n - K\sigma_n,\, z_n + K\sigma_n]\right) \quad \Leftrightarrow$$
$$\left(\|X\|_\infty = \max_{1\leq n\leq N} \|x_n\| \leq K\right)$$

The condition of integrity (1), that all real positions lie in the corresponding protection volume, translates to having the infinity norm of the vector X smaller than the K-factor. Thus we are interested in computing the probability of large deviations of a normalized Gaussian random variable with zero mean.

III. State of the art methods for K-factor determination

A. The MOPS method

The K-factor determination by the MOPS detailed in [2] is based on the following modelisation of the time correlation of the reduced errors: it assumes that $X(t)$ and $X(t+\delta t)$ are perfectly correlated if δt is smaller than the reference correlation time $T_0 = 360$ seconds and independent if δt is larger than 360 seconds. The numerical value of T_0 is motivated by the time scale involved in ionospheric corrections.

From here, the determination of the K-factor is a straightforward calculation. For a time window of length T, the MOPS model considers only $n_0 = \lceil T/T_0 \rceil$ independent measurements (one per time window of $T_0 = 360$ s). X denotes the vector of reduced errors and follows a n_0-dimensional centred Gaussian distribution of correlation matrix I_{n_0} (the identity matrix). Then we have for one-dimensional measurements, using the independence of the errors:

$$\mathbb{P}(\|X\|_\infty \leq K) = \prod_{k=1}^{n_0} \mathbb{P}(\|x_k\| \leq K) = F_1(K)^{n_0}$$

The probability factorises from the independence property. In the case $d \geq 2$, one has to replace the function F_1 by the corresponding function F_d given in Appendix A. In the one-dimensional case, we have $F_1(K) = \mathrm{erf}\left(K/\sqrt{2}\right)$ where erf denotes the error function[1]. Imposing the integrity condition (1) and using that in practise I_R is very small gives the K-factor formula:

$$K = F_1^{-1}\left((1-I_R)^{\frac{1}{n_0}}\right) \approx F_1^{-1}\left(1 - \frac{I_R}{n_0}\right) \tag{3}$$

where n_0 is the ceiling of the ratio T/T_0. This is the same formula as the one given in [2].

As a numerical illustration, we compute the K-factor for several aviation cases, with requirements described in [2]:

- for aviation precision approach (PA) and vertical protection level, we have $I_R = 10^{-7}$ and $d = 1$ for a time window of $T = 150$ s (thus $n_0 = 1$), which gives a K-factor of $K_{\mathrm{VPA}} = 5.33$
- for PA and horizontal protection level, we have $I_R = 2\times10^{-9}$, $d = 1$ (it covers only the along-track horizontal direction) and $T = 150$ s, which gives a K-factor of $K_{\mathrm{HPA}} = 6.0$
- for en route to non-precision approach (NPA), we have $I_R = 5 \times 10^{-8}$, $d = 2$ and $T = 3600$ s (thus $n_0 = 10$), which gives a K-factor of $K_{\mathrm{NPA}} = 6.18$

B. The GMK method

Recently, a novel method for the K-factor calculation has been published by [3]. The GMK method consists in comparing the reduced errors $X(t)$ to an autoregressive model of order 1 (AR(1) model) $Y(t)$ of stationary variance σ and parameter $e^{-\Delta t/\tau}$. Δt is the time step of the GNSS measurements (typically 1 s) and τ is the time constant of the exponential correlation decay. The parameters of the AR(1) process $Y(t)$ are chosen in order to impose the following condition on the power spectral density (PSD):

$$\forall f,\ S_X(f) \leq S_Y(f) \tag{4}$$

where S denotes the PSD of the processes X or Y. If the inequality is satisfied, we say that Y PSD-overbounds the process X.

The choice of the AR(1) model is motivated by the fact that the corresponding K-factor can be upper-bounded by considering its continuous version, the Ornstein-Uhlenbeck process. The result is:

$$\mathbb{P}\left(\max_t \|Y(t)\|_2 > K\right) \approx$$
$$\frac{1}{\Gamma\left(\frac{d}{2}\right)}\left(\frac{K^2}{2}\right)^{\frac{d}{2}} e^{-\frac{K^2}{2}}\left[\frac{2T}{\tau}\left(1 - \frac{d}{K^2}\right) + \frac{2}{K^2}\right]$$

where d denotes the dimension of the individual measurement. The K-factor is obtained by inverting the above formula and we can conclude that the K-factor of X is upper-bounded by:

$$K_X \leq \sigma K_Y(\tau) \tag{5}$$

For more details and proofs, we refer to the publication [3] and to the references therein.

IV. Limits of the state of the art methods

A. Limits of the MOPS model

It is straightforward to see that for the aviation VPA and HPA K-factor determination, the MOPS values are always a lower bound of the real K-factor for the condition (1) under the assumptions of Gaussian distribution of section II-A. Assuming one measurement per TTA, the vector of reduced errors X is a centred Gaussian vector of dimension $N = 25$. We denote by C its (known or unknown) correlation matrix.

[1]The error function is defined by the following integral:

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2} dt$$

It is linked to the normal cumulative distribution function Φ by the relation $\mathrm{erf}(x) = 2\Phi(x\sqrt{2}) - 1$.

Let us fix K_M the MOPS K-factor for aviation VPA or HPA corresponding the respective specified I_R. Then since $(\|X\|_\infty \leq K_M) \Rightarrow (\|x_1\| \leq K_M)$, we have:

$$\mathbb{P}(\|X\|_\infty \leq K_M) \leq \mathbb{P}(\|x_1\| \leq K_M) = 1 - I_R \qquad (6)$$

because the MOPS K-factor is computed with a unique one-dimensional Gaussian ($n_0 = \lceil T/T_0 \rceil = 1$ for $T = 150$ s) and x_1 follows a normal distribution of variance 1. This inequality is in the wrong direction compared to the integrity condition (1) and expresses the fact that the MOPS K-factor defines a rigorous protection volume only in the limiting case where all measurements are perfectly correlated (meaning that $\forall i, j,\ C_{ij} = 1$), which is precisely the assumption of the MOPS. This means that for short time intervals, the MOPS K-factor value of 5.33 corresponds to the most optimistic case under the assumptions of section II-A.

A more rigorous approach would be to calculate the number of effective number of independent samples as defined in [4]. This however requires a prior knowledge of the correlation matrix and extensive numerical calculations. This motivates us to turn to different methods in order to compute rigorously the K-factor.

B. Limits of the GMK method

The GMK method [3] has two main advantages compared to the MOPS method. First, it makes a finer modelisation of the time correlation of the reduced errors, and only assumes that X is a stationary series. Second, it gives a rigorous upper-bound of the K-factor over the hypothesis presented in section II-A. However, it also has some important limitations.

a) K-factor dependency on the shape of the distribution: One important drawback of the GMK method is the linear dependence on σ in formula (5). This makes the K-factor returned by the GMK method very sensitive to the value of σ. If the shape of the reduced errors PSD is too different from the AR(1) PSD, a large variance σ is needed for the process Y to PSD-overbound the process X. This in turn, generates a large K-factor.

As an illustrative example, we consider the following toy-model for the reduced error series $X(t)$. We consider that the positioning errors follow an AR(1) process with time constant $\tau = 360$ s for long time-scale error sources plus a small white noise (representing for example some thermal fluctuations at the user level). The error is dominated by the AR(1) process with variance $\sigma^2_{\mathrm{AR}(1)} = 0.95$ and the white noise has variance $\sigma_n^2 = 0.05$. This makes a process $X(t)$ with a PSD very similar to the PSD of a pure AR(1) but with a heavier tail (see figure 1). For the numerical example, we take the specifications of aviation APV: $T = 150$ s, TTA $= 6$ s (meaning that we consider one position estimate per TTA) and $I_R = 10^{-7}$.

For the pure AR(1) process with $\tau = 360$ s, the GMK method gives a factor $K = 5.80$, about 10% higher than the MOPS prescription of 5.33 for aviation APV. However with the addition of the white noise, the K-factor becomes $K = 9.86$ (and an optimal PSD-overbounding with a process Y of parameters $\sigma = 1.65$ and $\tau = 126$ s). This illustrates that a small deviation from an AR(1) process translates to a large inflation of the K-factor in the GMK method when doing the overbounding process.

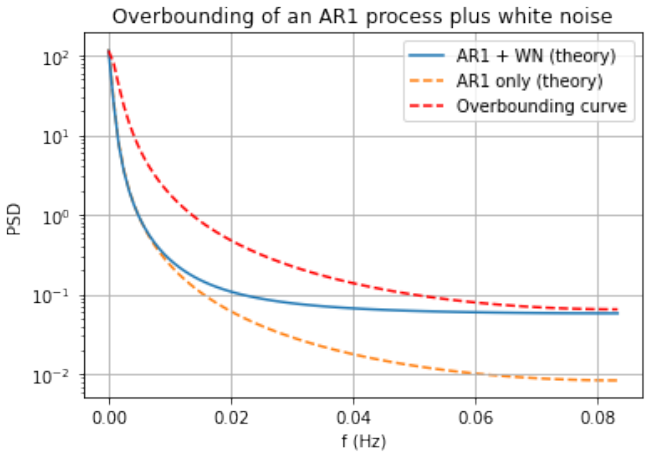

Fig. 1. Plot of the theoretical PSD of a process $X(t) = X_{\mathrm{AR1}}(t) + X_N(t)$ where X_{AR1} is an AR(1) process of variance 0.95 and time constant $\tau = 360$ s and X_N is a white noise of variance 0.05 (in blue continuous). In yellow dashed line, the PSD of an AR(1) process with the same time constant and variance normalized to 1. In red dashed line, the AR(1) process which PSD-overbounds $X(t)$ with parameters $\sigma = 1.65$ and $\tau = 126$ s.

b) Real-time determination of the K-factor: A question arisen by the GMK method is how to monitor in real-time the error correlation and the K-factor computation. In GNSS augmentation systems, the error correlation and PSD-overbounding are more naturally done at the pseudo-range level. However, even if this is done for each line of sight, it does not translate easily to a PSD-overbounding at position level. This is because the pseudo-range errors are multiplied by the geometry matrix which is varying in time as the GNSS satellites move across the sky and cross the horizon. Even if the PSD-overbounding is done at pseudo-range level, it is difficult to derive a general overbounding at position level independent of the user's position and has to be done for every user.

Furthermore, the errors monitored by GNSS augmentation systems are typically orbit and propagation errors and do not include user-specific errors such as multipath. These errors and their time correlation are strongly dependent on the user's environment and movement so their effects are difficult to include in the GMK method.

V. Proposed method: the K-factor in the independent case

In this section, we suggest to compute the K-factor by considering that all measurements are independent, meaning that we consider $N = T/\mathrm{TTA}$ effective independent samples. This method has the advantage of simplicity: the K-factor can be computed easily for any integrity risk and any number

K_I, $d=1$	10^{-3}	10^{-5}	10^{-7}	10^{-9}
1	3.291	4.417	5.327	6.109
10	3.890	4.892	5.731	6.467
25	4.107	5.069	5.884	6.604
150	4.504	5.400	6.174	6.865
3600	5.138	5.944	6.658	7.305
10800	5.341	6.122	6.818	7.451

K_I, $d=2$	10^{-3}	10^{-5}	10^{-7}	10^{-9}
1	3.717	4.799	5.678	6.438
10	4.292	5.257	6.070	6.786
25	4.500	5.428	6.219	6.920
150	4.882	5.749	6.501	7.174
3600	5.495	6.277	6.972	7.604
10800	5.691	6.450	7.128	7.747

TABLE I
NUMERICAL VALUE OF THE K-FACTOR COMPUTED IN THE INDEPENDENT CASE BY THE FORMULA (7) FOR DIFFERENT VALUE OF N ($= \lceil T/\mathrm{TTA} \rceil$) AND I_R IN THE CASE OF UNIDIMENSIONAL MEASUREMENTS. THE INDEPENDENT CASE IS AN UPPER-BOUND OF THE K-FACTOR FOR ANY CORRELATION MATRIX C AND FIXED I_R, d, N.

of estimates in the time-window. Moreover, we can prove mathematically that it is a rigorous upper-bound of the optimal K-factor, whatever the correlation structure of the successive measurements. Furthermore, it is also quite robust if certain assumptions of section II-A are relaxed. On the numerical side, the K-factor can never be smaller than the one obtained by the MOPS method but it is often numerically better than the factor obtained with the GMK method.

A. Computation of the K-factor in the independent case

We consider here the vector of reduced errors X of size N and assume that all measurements are independent: this means that X follows a Gaussian distribution of correlation matrix $C = I_N$ (the identity matrix). The calculation is the same than in the previous section, equation (3): the multiple integral factorises and we derive an analytic formula for the K-factor that depends only on N and the integrity risk I_R.

The independent K-factor formula : The expression for the K-factor in the independent case is given in general by the formula $K_I = F_d^{-1}\left((1-I_R)^{\frac{1}{N}}\right)$ where the function F_d is given explicitly in Appendix A. In the most common cases $d=1$ and $d=2$, they are explicitly:

$$K_I^{d=1} = \sqrt{2}\,\mathrm{erf}^{-1}\left((1-I_R)^{\frac{1}{N}}\right) \quad (7)$$

$$K_I^{d=2} = \sqrt{-2\log\left(1-(1-I_R)^{\frac{1}{N}}\right)}$$

where I_R is the integrity risk, N the number of position estimates in the time window and d the dimensionality of the position estimate.

The table I shows numerical values corresponding to different values of N and different level of integrity risk for $d=1$ and $d=2$. From the discussion in section II, N is given by the ceiling of the ratio T/TTA.

The main point of the K-factor in the independent case is that for fixed N the formula (7) gives a rigorous upper-bound of the optimal K-factor whatever the correlation matrix of the safety index vector X. In other words, the probability of having large deviations is maximum in the case where the correlation matrix is diagonal. The real measurement errors are correlated in an unknown way, but the correlation has an effect of decreasing the optimal K-factor associated to the measurement series. This result is summarized in the following theorem.

Theorem 1: Let X be a random vector of dimension N following a Gaussian distribution of zero mean and normalized correlation matrix C (meaning that for all diagonal elements are 1) and Y a random vector of same dimension with correlation matrix I_N. Then for all K we have the following inequality:

$$\mathbb{P}\left(\|X\|_\infty \leq K\right) \geq \mathbb{P}\left(\|Y\|_\infty \leq K\right) \quad (8)$$

In particular, for I_R and $K_I(I_R, N)$ given by formula (7), we have:

$$\mathbb{P}\left(\|X\|_\infty \leq K_I\right) \geq 1 - I_R$$

whatever the correlation matrix C.

A proof of this result in the case $d=1$ is presented in Appendix B-A following the method described in [5]. This result is known in the mathematical community since the 1960s, see for example [6]–[8]. Note that the hypothesis of Gaussian distribution in section II-A is necessary for the theorem to hold. In the case $d \geq 2$, a similar theorem still holds but the proof requires a stronger theorem, the Gaussian Correlation inequality proven in 2014 by [9] (see Appendix A and B-B).

Beyond the independent K-factor: For a fixed N and I_R, if C denotes the correlation matrix of the estimators and $K_C(I_R)$ is the optimal K-factor (the I_R quantile of a Gaussian distribution $\mathcal{N}(0, C)$), theorem 1 gives an upper bound of the K-factor. On the other hand, equation (6) gives a lower bound for the optimal K-factor so we have for all C:

$$\sqrt{2}\,\mathrm{erf}^{-1}(1-I_R) \leq K_C(I_R) \leq \sqrt{2}\,\mathrm{erf}^{-1}\left((1-I_R)^{\frac{1}{N}}\right) \quad (9)$$

Both bounds are optimal in the sense that they are reached for specific correlation matrices ($C = I_N$ for the upper bound and C a matrix of 1's for the lower bound).

If C can be estimated, one can compute the optimal K-factor $K_C(I_R)$ by extensive simulations. However, the numerical improvement is bounded by the left hand side of the inequality above (for $I_R = 10^{-7}$, $N = 25$ and $d = 1$, we can go from $K_I = 5.88$ to, in the most correlated case, $K = 5.33$). Moreover, the difficulty is to make sure that the computed K-factor is indeed an upper bound of the true K-factor because of possible errors in the determination of C.

B. Robustness of the independent method

Measurement campaigns have shown that the distribution of positioning errors typically have a heavier tail than a Gaussian distribution, contradicting our assumption in section II-A. The way to deal with non-Gaussian distributions is usually through CDF-overbounding [10], [11].

In this paragraph, we relax the assumption of the errors following a Gaussian distribution. Instead, we allow for any distribution, but we assume that for any measurement n, the marginal distribution of the error $z_n - z_n^*$ (marginalizing over all other measurements) is correctly CDF-overbounded by a Gaussian distribution with variance σ_n up to some value $1 - L_B$ with L_B very small. In other words, the GNSS system has some information on the error distribution of each measurement. This CDF-overbounding defined in [10], [11] means that for all n, we have:

$$\mathbb{P}\left(\|x_n\| \leq k\right) \geq \mathbb{P}\left(\|y\| \leq k\right) = F_1(k)$$

where $y \sim \mathcal{N}(0,1)$ for all k as long as the right-hand side $F_1(k)$ is smaller than $1 - L_B$. This is because for distributions with a heavier tail than a Gaussian, CDF-overbounding by a Gaussian on the entire real line is not possible. Note that the variance σ_n of the overbounding Gaussian can be very dependent on the limit L_B.

Now since X is not a Gaussian vector, we cannot use theorem 1, however we can use the following inequality[2]

$$\mathbb{P}\left(\max_n \|x_n\| \leq k\right) \geq 1 - \sum_n \mathbb{P}\left(\|x_n\| > k\right) \qquad (10)$$

If we choose $K = F_1^{-1}(1 - I_R/N)$ and assume that the CDF-overbounding condition holds (i.e. $L_B \leq I_R/N$), we have $\mathbb{P}\left(\|x_n\| > k\right) \leq I_R/N$ and so:

$$\mathbb{P}\left(\max_n \|x_n\| \leq K\right) \geq 1 - N\frac{I_R}{N} = 1 - I_R \qquad (11)$$

This is precisely the condition of integrity (1). It is met at the conditions that each error is CDF-overbounded until $L_B = I_R/N$ and for a K-factor equal to:

$$K = F_1^{-1}(1 - I_R/N) \approx F_1^{-1}\left((1 - I_R)^{\frac{1}{N}}\right) = K_I(I_R, N)$$

neglecting only terms of order I_R^2 and higher. This means that the K-factor computed by the independent method can be extended to the case where the error distribution is non-Gaussian. The weaker condition is to have a Gaussian CDF-overbounding of the error distribution to a level of I_R/N. Then $K_I(I_R, N)$ (7) remains a rigorous upper-bound of the optimal K-factor up to terms quadratic in I_R.

Note that CDF-overbounding also includes the case where the standard deviation is overestimated in the case of Gaussian distributions. This is because a Gaussian with a larger standard deviation always CDF-overbounds a Gaussian with smaller standard deviation for all L_B. Thus the K-factor in the independent method is robust in the case where the σ_n are bounded from above instead of being accurately estimated.

[2]It is obtained by applying $\mathbb{P}\left(A \cap B\right) = 1 - \mathbb{P}\left(\bar{A} \cup \bar{B}\right) \geq 1 - \mathbb{P}\left(\bar{A}\right) - \mathbb{P}\left(\bar{B}\right)$ with $A = \{X : |x_1| \leq k\}$ and $B = \{X : |x_2| \leq k\}$ and then induction over n.

C. Numerical comparison with the MOPS and GMK method

Let us highlight the numerical difference between the K-factor in several cases specified by the MOPS.

- for aviation APV ($T = 150$ s, TTA $= 6$ s leading to $N = 25$ and $I_R = 10^{-7}$) the computation with the independent method gives $K_{I,\mathrm{VPA}} = 5.88$, corresponding to a 10% increase of the protection volume compared to the MOPS value of 5.33.
- for aviation APH ($I_R = 2 \times 10^{-9}$, $d = 1$ and $N = 25$), we get $K_{I,\mathrm{HPA}} = 6.50$, which is a 8% increase compared to the MOPS value of 6.0.
- for en-route to NPA ($I_R = 5 \times 10^{-8}$, $d = 2$, $T = 1$ h and TTA $= 10$ s so $N = 360$), the factor in the independent case is $K_{I,\mathrm{NPA}} = 6.74$, a 9% increase compared to the MOPS value.

In all cases, the independent K-factor is larger than the MOPS values because it considers more than one independent measurement per T_0. However the increase is reasonable considering that the independent K-factor satisfies the integrity condition (1).

Compared to the GMK method, the K-factor in the independent case has the advantage of not having a linear dependence on one parameter. The independent K-factor grows with the number of measurements N but the growth is only logarithmic. Since the independent computation neglects the correlation completely, we expect the GMK method to perform better for large correlation time. On the other hand, the GMK method is limited by the rapid growth of the K-factor with σ: it is adapted only for PSD profiles close to the shape of an AR(1) process. To illustrate, we go back to the toy-model of figure 1 for which the GMK method gives a K-factor of 5.80 (in a pure AR(1) case) and 9.86 (AR(1) plus white noise). The independent K-factor is 5.88 in both cases, slightly higher than the AR(1) case, but significantly smaller than the perturbed AR(1) case. In practise, the independent K-factor is often smaller than the GMK factor, and moreover is easier to implement.

VI. Conclusion

In conclusion, we have presented the formula (7) (and (13) in the general case) for K-factor computations based on a majoration by the independent case. The formula is simple, analytic, and depends only on the specifications of the integrity risk I_R, the number of measurements N in the time window of interest and the dimension d of the protection volume. It requires no additional assumption on the correlation structure of the successive position estimators which is difficult to measure in practice and can vary wildly in different environments and conditions.

Theorem 1 (and 2 in the general case) ensures that the resulting K-factor satisfies mathematically the simplified integrity condition (1) under the assumptions of section II-A.

Section V-B shows that the integrity condition still holds with weaker assumptions, namely that the estimator error is correctly CDF-overbounded by a Gaussian distribution up to the I_R/N quantile.

Compared to previously existing methods, the independent calculation has the advantage of being mathematically justified by theorems 1 and 2. Because the proposed method is simple, there is no need to estimate and overbound the correlation of the time series, which makes the method less sensitive to mismodelling of the error time series. Moreover, this approach is robust to deviations from the Gaussian assumption. The obtained numerical values are larger than the ones obtained with the MOPS method – it is the price to pay for mathematical rigor – but often tighter than the ones obtained with the GMK method. Because of its simplicity, mathematical validity and robustness, we recommend to use the independent K-factor formula for new integrity services like ARAIM and SBAS extension beyond aviation.

APPENDIX A
GENERALISATION TO HIGHER DIMENSIONS

In the main text, we have treated the measurements as one-dimensional but GNSS position measurements are often of higher dimension. All of the presented results easily generalise to higher d with small modifications.

A. Definition of the protection volume

If $Z = (z_n)_{1\leq n\leq N}$ is the series of position estimators during the time window of interest with $z_n \in \mathbb{R}^d$, it follows by assumption a Gaussian distribution centred on the true positions and with a square covariance matrix Σ of dimension Nd. The "error" σ_n^2 of the estimator n is defined as the largest eigenvalue of the covariance matrix $\mathrm{cov}(z_n, z_n)$ of dimension $d \times d$ (see [1] appendix J). From this, the protection volume V_n definition (2) is replaced by a ball (for the usual Euclidean norm) centred on the estimator output z_n and of radius $K\sigma_n$:

$$V_n = B(z_n, K\sigma_n) = \left\{z \in \mathbb{R}^d \,:\, \|z - z_n\|_2 \leq K\sigma_n\right\}$$

Then the condition of integrity (the true position is always in the protection volume for the whole time window) translates to the vector X as:

$$\left(\forall n \in [\![1, N]\!],\ z_n^* \in V_n\right) \Leftrightarrow \left(\max_{1\leq n\leq N} \|x_n\|_2 \leq K\right)$$

This is some sort of infinity norm of the vector X that we will denote as $\|X\|_\infty$. The reduced errors X are normalized in the sense that the covariance matrix $\mathrm{cov}(x_n, x_n)$ has eigenvalues as most 1.

Now that we have defined the protection volume, we want to determine an upper-bound to the K-factor in order to ensure the integrity condition $\mathbb{P}(\|X\|_\infty \leq K) \geq 1 - I_R$. As in the $d = 1$ case, such an upper-bound is given by the case where all estimators are independent and furthermore spatially homogeneous. The theorem 1 is restated as:

Theorem 2: Let $X = (x_n)_{1\leq n\leq N}$ be a random vector of dimension Nd following a Gaussian distribution of zero mean and normalized covariance matrix in the sense that for all n, the matrix $\mathrm{cov}(x_n, x_n)$ has eigenvalues as most 1. Let $Y = (y_n)_{1\leq n\leq N}$ be a random vector of same dimension with covariance $\mathrm{cov}(y_n, y_n) = I_d$ for all n and $\mathrm{cov}(y_n, y_m) = 0$ if $n \neq m$ (all estimators are independent and spatially homogeneous). Then for all K we have the following inequality:

$$\mathbb{P}\left(\|X\|_\infty \leq K\right) \geq \mathbb{P}\left(\|Y\|_\infty \leq K\right) \tag{12}$$

where $\|X\|_\infty = \max(\|x_n\|_2)$. In particular, for I_R and $K_I(I_R, N)$ given by formula (7), we have:

$$\mathbb{P}\left(\|X\|_\infty \leq K_I\right) \geq 1 - I_R$$

whatever the covariance matrix of X.

B. Explicit independent K-factor formula

The K-factor in the independent case is easy to calculate. We have $\mathbb{P}(\|X\|_\infty \leq K) = \prod \mathbb{P}(\|x_n\| \leq K) = F_d(K)^N$ where the function F_d is given explicitly by the expression:

$$F_d(K) = \int_{\|x\|\leq K} \frac{1}{(2\pi)^{\frac{d}{2}}} e^{-\frac{1}{2}\|x\|^2} d^d x$$

It is the CDF of the Euclidean norm of a Gaussian random vector in dimension d of zero mean and correlation matrix I_d. Then, the K-factor in the independent case is given by

$$K_I(I_R, N) = F_d^{-1}\left((1 - I_R)^{\frac{1}{N}}\right) \tag{13}$$

For small d, the explicit forms are $F_1(K) = \mathrm{erf}\left(K/\sqrt{2}\right)$ and $F_2(K) = 1 - e^{-\frac{K^2}{2}}$ with the independent K-factor formula given in equation (7). For $d \geq 3$, the function F_d can be expressed as a combination of polynomials, exponential and error function but cannot be analytically inverted:

$$F_3(K) = \mathrm{erf}\left(\frac{K}{\sqrt{2}}\right) - \sqrt{\frac{2}{\pi}} K e^{-\frac{K^2}{2}}$$

To finish, the lower and upper bounds of equation (9) generalise in dimension d to:

$$\sqrt{2}\,\mathrm{erf}^{-1}(1 - I_R) \leq K(I_R, C) \leq F_d^{-1}\left((1 - I_R)^{\frac{1}{N}}\right)$$

C. CDF-overbounding

For $d \geq 2$, there is no universal definition of CDF-overbounding. However, all the developments of section V-B hold by replacing the overbounding condition by the following condition: for all k such as $F_d(k) \leq 1 - L_B$ we have:

$$\mathbb{P}\left(\|x_n\| \leq k\right) \geq F_d(k)$$

meaning that the first $1 - L_B$ fraction of the error distribution is "less spread out" than a Gaussian distribution.

APPENDIX B
PROOF OF THE THEOREM

A. $d = 1$ *case*

In this section, we will sketch the proof of theorem 1 in the case $d = 1$. The theorem comes from the following result proven by Kathri in 1967. Let $Y = (Y_1, Y_2)$ be a random Gaussian vector of dimension $n = n_1 + n_2$ (with Y_i of size n_i) of zero mean and covariance matrix

$$\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12}^T & \Sigma_{22} \end{pmatrix}$$

with the partition corresponding to Y_1 and Y_2. Let C_1 and C_2 be two symmetric convex subsets of $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$ respectively. If $\mathrm{rank}(\Sigma_{12}) = 1$, we have the following inequality:

$$\mathbb{P}(Y_1 \in C_1 \text{ and } Y_2 \in C_2) \geq \mathbb{P}(Y_1 \in C_1)\,\mathbb{P}(Y_2 \in C_2) \quad (14)$$

Then theorem 1 follows from applying the above result with $n_1 = 1$, $n_2 = n - 1$, $C_1 = \{|y_1| \leq K\}$, $C_2 = \{\forall i \geq 2, |y_i| \leq K\}$ and then induction on n. With this partitioning, Σ_{12} has size $1 \times (n-1)$ and has thus rank at most 1 so (14) applies.

Let us now move to the proof of (14). Since $\mathrm{rank}(\Sigma_{12}) = 1$, there exists two vectors $a_1 \in \mathbb{R}^{n_1}$ and $a_2 \in \mathbb{R}^{n_2}$ such that $\Sigma_{12} = a_1 a_2^T$. We introduce a new normal random vector $\tilde{Y} = (Y_1, Y_2, Z)$ of dimension $n + 1$ with zero mean and covariance matrix:

$$\tilde{\Sigma} = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} & a_1 \\ \Sigma_{12}^T & \Sigma_{22} & a_2 \\ a_1^T & a_2^T & 1 \end{pmatrix}$$

Then the distribution of (Y_1, Y_2) conditional on $Z = z$ is a multivariate normal of mean $\bar{\mu}$ and covariance $\overline{\Sigma}$ where $\bar{\mu} = (za_1, za_2)^T$ and

$$\overline{\Sigma} = \Sigma - \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}^T = \begin{pmatrix} \Sigma_{11} - a_1 a_1^T & 0 \\ 0 & \Sigma_{22} - a_2 a_2^T \end{pmatrix}$$

and therefore Y_1 and Y_2 are conditionally independent given Z (note that this is only possible because $\Sigma_{12} = a_1 a_2^T$ thus Σ_{12} must have rank at most 1).

Now we define the two functions of z:

$$g_i(z) = \mathbb{P}(Y_i \in C_i | Z = z) = \int_{C_i - z a_i} f_i(y) dy$$

for $i = 1, 2$, where f_i denotes the Gaussian density function of covariance $\Sigma_{ii} - a_i a_i^T$. We use Anderson's theorem (1955) [12] to show that the two functions g_i are symmetric and non-increasing functions of $|z|$, meaning that:

$$|z_1| \leq |z_2| \Rightarrow g_i(z_1) \geq g_i(z_2) \quad (15)$$

More precisely, Anderson's theorem states that for $f : \mathbb{R}^n \to \mathbb{R}$ a nonnegative symmetric unimodal function (unimodal means here that for all real c, the set $\{x \in \mathbb{R}^n : f(x) \geq c\}$ is convex), K a symmetric convex subset of $\mathbb{R}^n$, then for every $y \in \mathbb{R}^n$ the function

$$h : t \mapsto \int_{K + ty} f(x) dx$$

is a symmetric unimodal function of t. For a proof, we refer to [5], [12]. It is straightforward to check that the hypothesis of Anderson's theorem are verified for the functions g_i and that a symmetric and unimodal function of $\mathbb{R}$ satisfies the property (15).

From the property (15), we can write:

$$\forall z_1, z_2,\ (g_1(z_1) - g_1(z_2))\,(g_2(z_1) - g_2(z_2)) \geq 0$$

and thus taking the expectation with respect to z_1 then z_2, we see that $\mathrm{cov}(g_1(Z), g_2(Z)) \geq 0$. But we have that:

$$E[g_1(Z)] = E[\mathbb{P}(Y_1 \in C_1 | Z)] = \mathbb{P}(Y_1 \in C_1)$$

and similarly $E[g_1(Z)] = \mathbb{P}(Y_1 \in C_1)$ whereas

$$\begin{aligned} E[g_1(Z) g_2(Z)] &= E[\mathbb{P}(Y_1 \in C_1 | Z)\,\mathbb{P}(Y_2 \in C_2 | Z)] \\ &= E[\mathbb{P}(Y_1 \in C_1 \text{ and } Y_2 \in C_2 | Z)] \\ &= \mathbb{P}(Y_1 \in C_1 \text{ and } Y_2 \in C_2) \end{aligned}$$

by the conditional independence of Y_1, Y_2 given Z. So the positive correlation $\mathrm{cov}(g_1(Z), g_2(Z)) \geq 0$ gives us exactly equation (14) which concludes the proof.

B. $d \geq 2$ *case*

If $d \geq 2$, the proof presented above fails because the condition $\mathrm{rank}(\Sigma_{12}) = 1$ can no longer be insured. The result remains true and is a consequence of the following stronger and more recent theorem:

> *Theorem 3 (Gaussian Correlation Inequality):* Let X be a Gaussian vector of dimension n of zero mean. Then for all convex sets $E, F \subset \mathbb{R}^n$ that are symmetric about the origin, we have the inequality
>
> $$\mathbb{P}(X \in E \cap F) \geq \mathbb{P}(X \in E)\,\mathbb{P}(X \in F)$$

For a proof of this theorem which is beyond the scope of this paper, we refer to the articles [9], [13].

Now let X, Y two Gaussian vectors of $\mathbb{R}^{Nd}$ verifying the conditions of theorem 2. For a fixed K we define the sets $E_n = \left\{X \in \mathbb{R}^{Nd} : \|x_n\|_2 \leq K\right\}$ which are symmetric about the origin and convex. By successive applications of the Gaussian Correlation Inequality, we get

$$\begin{aligned} \mathbb{P}(\|X\|_\infty \leq K) &= \mathbb{P}\left(X \in \bigcap_{n=1}^N E_n\right) \\ &\geq \prod_{n=1}^N \mathbb{P}(X \in E_n) = \prod_{n=1}^N \mathbb{P}(\|x_n\|_2 \leq K) \end{aligned}$$

Then for all n we have $\mathbb{P}(\|x_n\|_2 \leq K) \geq \mathbb{P}(\|y_n\|_2 \leq K)$. This is straightforward to see because the covariance matrix $\mathrm{cov}(x_n, x_n)$ has eigenvalues at most one and $y_n \sim \mathcal{N}(0, I_d)$. This finishes the proof:

$$\begin{aligned} \mathbb{P}(\|X\|_\infty \leq K) &\geq \prod_{n=1}^N \mathbb{P}(\|x_n\|_2 \leq K) \\ &\geq \prod_{n=1}^N \mathbb{P}(\|y_n\|_2 \leq K) = \mathbb{P}(\|Y\|_\infty \leq K) \end{aligned}$$

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