# Technical Note: Mathematical Development of the Interplexing 6 Signals for the Galileo E1 Band

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# Mathematical Development of the Interplexing 6 Signals for the Galileo E1 Band

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The general expression of the Interplexing 6 signals is:

$$\begin{aligned} s(t) &= \sqrt{2P}\cos\Big(2\pi f_c t - \frac{\pi}{2}s_1(t) + \beta_2 s_1(t)s_2(t) + \beta_3 s_1(t)s_3(t) + \beta_4 s_1(t)s_4(t) + \beta_5 s_1(t)s_5(t) + \beta_6 s_1(t)s_6(t) + \phi\Big) \\ &= \sqrt{2P}\cos\left(2\pi f_c t + A + B + C + D + E + \phi\right) \end{aligned} \tag{1}$$

where:

- $A = \beta_2 s_1(t)s_2(t)$
- $B = \beta_3 s_1(t)s_3(t)$
- $C = \beta_4 s_1(t)s_4(t)$
- $D = \beta_5 s_1(t)s_5(t)$
- $E = \beta_6 s_1(t)s_6(t)$

then:

$$s(t) = \sqrt{2P}\begin{bmatrix} \cos(2\pi f_c t + \phi)\begin{bmatrix}\cos\left(-\frac{\pi}{2}s_1(t)\right)\cos\left(A+B+C+D+E\right) - \\ \sin\left(-\frac{\pi}{2}s_1(t)\right)\sin\left(A+B+C+D+E\right)\end{bmatrix} \\ +\sin(2\pi f_c t + \phi)\begin{bmatrix}\sin\left(-\frac{\pi}{2}s_1(t)\right)\cos\left(A+B+C+D+E\right) + \\ \cos\left(-\frac{\pi}{2}s_1(t)\right)\sin\left(A+B+C+D+E\right)\end{bmatrix}\end{bmatrix} \tag{2}$$

where $\cos\left(-\frac{\pi}{2}s_1(t)\right) = 0$, yielding $s(t)$ to:

$$s(t) = \sqrt{2P}\begin{bmatrix} \cos(2\pi f_c t + \phi)\left[s_1(t)\sin\left(A+B+C+D+E\right)\right] \\ +\sin(2\pi f_c t + \phi)\left[s_1(t)\cos\left(A+B+C+D+E\right)\right]\end{bmatrix} \tag{3}$$

where $\sin\left(A+B+C+D+E\right)$ is:

$$\sin\left(A+B+C+D+E\right) = \begin{bmatrix} \sin(A)\cos(B)\cos(C)\cos(D)\cos(E) - \sin(A)\cos(B)\cos(C)\sin(D)\sin(E) - \\ \sin(A)\cos(B)\sin(C)\sin(D)\cos(E) - \sin(A)\cos(B)\sin(C)\cos(D)\sin(E) - \\ \sin(A)\sin(B)\sin(C)\cos(D)\cos(E) + \sin(A)\sin(B)\sin(C)\sin(D)\sin(E) - \\ \sin(A)\sin(B)\cos(C)\sin(D)\cos(E) - \sin(A)\sin(B)\cos(C)\cos(D)\sin(E) + \\ \cos(A)\sin(B)\cos(C)\cos(D)\cos(E) - \cos(A)\sin(B)\cos(C)\sin(D)\sin(E) - \\ \cos(A)\sin(B)\sin(C)\sin(D)\cos(E) - \cos(A)\sin(B)\sin(C)\cos(D)\sin(E) + \\ \cos(A)\cos(B)\sin(C)\cos(D)\cos(E) - \cos(A)\cos(B)\sin(C)\sin(D)\sin(E) + \\ \cos(A)\cos(B)\cos(C)\sin(D)\cos(E) + \cos(A)\cos(B)\cos(C)\cos(D)\sin(E) \end{bmatrix} \tag{4}$$

and

$\cos(A+B+C+D+E)$ is:

$$\cos(A+B+C+D+E) = \begin{bmatrix} \cos(A)\cos(B)\cos(C)\cos(D)\cos(E) - \cos(A)\cos(B)\cos(C)\sin(D)\sin(E) - \\ \cos(A)\cos(B)\sin(C)\sin(D)\cos(E) - \cos(A)\cos(B)\sin(C)\cos(D)\sin(E) - \\ \cos(A)\sin(B)\sin(C)\cos(D)\cos(E) + \cos(A)\sin(B)\sin(C)\sin(D)\sin(E) - \\ \cos(A)\sin(B)\cos(C)\sin(D)\cos(E) - \cos(A)\sin(B)\cos(C)\cos(D)\sin(E) - \\ \sin(A)\sin(B)\cos(C)\cos(D)\cos(E) + \sin(A)\sin(B)\cos(C)\sin(D)\sin(E) + \\ \sin(A)\sin(B)\sin(C)\sin(D)\cos(E) + \sin(A)\sin(B)\sin(C)\cos(D)\sin(E) - \\ \sin(A)\cos(B)\sin(C)\cos(D)\cos(E) + \sin(A)\cos(B)\sin(C)\sin(D)\sin(E) - \\ \sin(A)\cos(B)\cos(C)\sin(D)\cos(E) - \sin(A)\cos(B)\cos(C)\cos(D)\sin(E) \end{bmatrix} \tag{5}$$

Then, from equation (3) and considering the definition of $A$,$B$,$C$,$D$,$E$, we derive:

$$s_1(t)\sin(A+B+C+D+E) = \begin{bmatrix} s_2(t)\sin(\beta_2)\cos(\beta_3)\cos(\beta_4)\cos(\beta_5)\cos(\beta_6) - \\ s_2(t)s_5(t)s_6(t)\sin(\beta_2)\cos(\beta_3)\cos(\beta_4)\sin(\beta_5)\sin(\beta_6) - \\ s_2(t)s_4(t)s_5(t)\sin(\beta_2)\cos(\beta_3)\sin(\beta_4)\sin(\beta_5)\cos(\beta_6) - \\ s_2(t)s_4(t)s_6(t)\sin(\beta_2)\cos(\beta_3)\sin(\beta_4)\cos(\beta_5)\sin(\beta_6) - \\ s_2(t)s_3(t)s_4(t)\sin(\beta_2)\sin(\beta_3)\sin(\beta_4)\cos(\beta_5)\cos(\beta_6) + \\ s_2(t)s_3(t)s_4(t)s_5(t)s_6(t)\sin(\beta_2)\sin(\beta_3)\sin(\beta_4)\sin(\beta_5)\sin(\beta_6) - \\ s_2(t)s_3(t)s_5(t)\sin(\beta_2)\sin(\beta_3)\cos(\beta_4)\sin(\beta_5)\cos(\beta_6) - \\ s_2(t)s_3(t)s_6(t)\sin(\beta_2)\sin(\beta_3)\cos(\beta_4)\cos(\beta_5)\sin(\beta_6) + \\ s_3(t)\cos(\beta_2)\sin(\beta_3)\cos(\beta_4)\cos(\beta_5)\cos(\beta_6) - \\ s_3(t)s_5(t)s_6(t)\cos(\beta_2)\sin(\beta_3)\cos(\beta_4)\sin(\beta_5)\sin(\beta_6) - \\ s_3(t)s_4(t)s_5(t)\cos(\beta_2)\sin(\beta_3)\sin(\beta_4)\sin(\beta_5)\cos(\beta_6) - \\ s_3(t)s_4(t)s_6(t)\cos(\beta_2)\sin(\beta_3)\sin(\beta_4)\cos(\beta_5)\sin(\beta_6) + \\ s_4(t)\cos(\beta_2)\cos(\beta_3)\sin(\beta_4)\cos(\beta_5)\cos(\beta_6) - \\ s_4(t)s_5(t)s_6(t)\cos(\beta_2)\cos(\beta_3)\sin(\beta_4)\sin(\beta_5)\sin(\beta_6) + \\ s_5(t)\cos(\beta_2)\cos(\beta_3)\cos(\beta_4)\sin(\beta_5)\cos(\beta_6) + \\ s_6(t)\cos(\beta_2)\cos(\beta_3)\cos(\beta_4)\cos(\beta_5)\sin(\beta_6) \end{bmatrix} \tag{6}$$

Let us now redefine the Galileo E1 OS equations in [1] as a function of $s_2(t)$, $s_3(t)$, $s_4(t)$ and $s_5(t)$:

$$\begin{gathered} s_{E1-OS_{data}}(t) = d_{E1-B}(t)c_{E1-B}(t)(\alpha BOC(1,1) + \beta BOC(6,1)) = \alpha s_2(t) + \beta s_4(t). \\ s_{E1-OS_{pilot}}(t) = c_{E1-C}(t)(\alpha BOC(1,1) - \beta BOC(6,1)) = \alpha s_3(t) - \beta s_5(t). \end{gathered} \tag{7}$$

considering equation (7), equation (4) can be simplified to :

$$\begin{bmatrix} s_2(t)\left[\sin(\beta_2)\cos(\beta_3)\cos(\beta_4)\cos(\beta_5)\cos(\beta_6) - \cos(\beta_2)\sin(\beta_3)\sin(\beta_4)\sin(\beta_5)\cos(\beta_6)\right] \\ +s_2(t)s_5(t)s_6(t)\left[-\sin(\beta_2)\cos(\beta_3)\cos(\beta_4)\sin(\beta_5)\sin(\beta_6) - \cos(\beta_2)\sin(\beta_3)\sin(\beta_4)\cos(\beta_5)\sin(\beta_6)\right] \\ +s_3(t)\left[\cos(\beta_2)\sin(\beta_3)\cos(\beta_4)\cos(\beta_5)\cos(\beta_6) - \sin(\beta_2)\cos(\beta_3)\sin(\beta_4)\sin(\beta_5)\cos(\beta_6)\right] \\ +s_2(t)s_4(t)s_6(t)\left[-\sin(\beta_2)\cos(\beta_3)\sin(\beta_4)\cos(\beta_5)\sin(\beta_6) - \cos(\beta_2)\sin(\beta_3)\cos(\beta_4)\sin(\beta_5)\sin(\beta_6)\right] \\ +s_5(t)\left[\cos(\beta_2)\cos(\beta_3)\cos(\beta_4)\sin(\beta_5)\cos(\beta_6) - \sin(\beta_2)\sin(\beta_3)\sin(\beta_4)\cos(\beta_5)\cos(\beta_6)\right] \\ +s_6(t)\left[\cos(\beta_2)\cos(\beta_3)\cos(\beta_4)\cos(\beta_5)\sin(\beta_6) + \sin(\beta_2)\sin(\beta_3)\sin(\beta_4)\sin(\beta_5)\sin(\beta_6)\right] \\ +s_4(t)\left[\cos(\beta_2)\cos(\beta_3)\sin(\beta_4)\cos(\beta_5)\cos(\beta_6) - \sin(\beta_2)\sin(\beta_3)\cos(\beta_4)\sin(\beta_5)\cos(\beta_6)\right] \\ +s_2(t)s_3(t)s_6(t)\left[-\sin(\beta_2)\sin(\beta_3)\cos(\beta_4)\cos(\beta_5)\sin(\beta_6) - \cos(\beta_2)\cos(\beta_3)\sin(\beta_4)\sin(\beta_5)\sin(\beta_6)\right] \end{bmatrix} \tag{8}$$

Consequently:

$$\begin{gathered} (\sin(\beta_2)\cos(\beta_3)\cos(\beta_4)\cos(\beta_5)\cos(\beta_6) - \cos(\beta_2)\sin(\beta_3)\sin(\beta_4)\sin(\beta_5)\cos(\beta_6)) = \alpha \\ (\cos(\beta_2)\sin(\beta_3)\cos(\beta_4)\cos(\beta_5)\cos(\beta_6) - \sin(\beta_2)\cos(\beta_3)\sin(\beta_4)\sin(\beta_5)\cos(\beta_6)) = \alpha \\ (\cos(\beta_2)\cos(\beta_3)\cos(\beta_4)\sin(\beta_5)\cos(\beta_6) - \sin(\beta_2)\sin(\beta_3)\sin(\beta_4)\cos(\beta_5)\cos(\beta_6)) = -\beta \\ (cos(\beta_2)\cos(\beta_3)\sin(\beta_4)\cos(\beta_5)\cos(\beta_6) - \sin(\beta_2)\sin(\beta_3)\sin(\beta_4)\sin(\beta_5)\sin(\beta_6)) = \beta \end{gathered} \tag{9}$$

This linear system lead to $\beta_2 = \beta_3$ and $\beta_4 = -\beta_5$, leaving equation (6) to:

$$s_1(t)\sin(A+B+C+D+E)=\begin{bmatrix} (s_2(t)+s_3(t))\cos(\beta_6)\frac{\sin(2\beta_2)}{2}+ \\ (s_4(t)-s_5(t))\cos(\beta_6)\frac{\sin(2\beta_4)}{2}+ \\ s_6(t)\sin(\beta_6)\frac{[\cos(2\beta_2)+\cos(2\beta_4)]}{2}+ \\ s_2(t)s_3(t)s_6(t)\sin(\beta_6)\left[\cos^2(\beta_2)\sin^2(\beta_4)-\sin^2(\beta_2)\cos^2(\beta_4)\right] \end{bmatrix} \tag{10}$$

Now, from equation (3) and considering the definition of $A$,$B$,$C$,$D$,$E$, we derive:

$$s_1(t)\cos(A+B+C+D+E)=\begin{bmatrix} s_1(t)\cos(\beta_2)\cos(\beta_3)\cos(\beta_4)\cos(\beta_5)\cos(\beta_6)- \\ s_1(t)s_5(t)s_6(t)\cos(\beta_2)\cos(\beta_3)\cos(\beta_4)\sin(\beta_5)\sin(\beta_6)- \\ s_1(t)s_4(t)s_5(t)\cos(\beta_2)\cos(\beta_3)\sin(\beta_4)\sin(\beta_5)\cos(\beta_6)- \\ s_1(t)s_4(t)s_6(t)\cos(\beta_2)\cos(\beta_3)\sin(\beta_4)\cos(\beta_5)\sin(\beta_6)- \\ s_1(t)s_3(t)s_4(t)\cos(\beta_2)\sin(\beta_3)\sin(\beta_4)\cos(\beta_5)\cos(\beta_6)+ \\ s_1(t)s_3(t)s_4(t)s_5(t)s_6(t)\cos(\beta_2)\sin(\beta_3)\sin(\beta_4)\sin(\beta_5)\sin(\beta_6)- \\ s_1(t)s_3(t)s_5(t)\cos(\beta_2)\sin(\beta_3)\cos(\beta_4)\sin(\beta_5)\cos(\beta_6)- \\ s_1(t)s_3(t)s_6(t)\cos(\beta_2)\sin(\beta_3)\cos(\beta_4)\cos(\beta_5)\sin(\beta_6)- \\ s_1(t)s_2(t)s_3(t)\sin(\beta_2)\sin(\beta_3)\cos(\beta_4)\cos(\beta_5)\cos(\beta_6)+ \\ s_1(t)s_2(t)s_3(t)s_5(t)s_6(t)\sin(\beta_2)\sin(\beta_3)\cos(\beta_4)\sin(\beta_5)\sin(\beta_6)+ \\ s_1(t)s_2(t)s_3(t)s_4(t)s_5(t)\sin(\beta_2)\sin(\beta_3)\sin(\beta_4)\sin(\beta_5)\cos(\beta_6)+ \\ s_1(t)s_2(t)s_3(t)s_4(t)s_6(t)\sin(\beta_2)\sin(\beta_3)\sin(\beta_4)\cos(\beta_5)\sin(\beta_6)- \\ s_1(t)s_2(t)s_4(t)\sin(\beta_2)\cos(\beta_3)\sin(\beta_4)\cos(\beta_5)\cos(\beta_6)+ \\ s_1(t)s_2(t)s_4(t)s_5(t)s_6(t)\sin(\beta_2)\cos(\beta_3)\sin(\beta_4)\sin(\beta_5)\sin(\beta_6)- \\ s_1(t)s_2(t)s_5(t)\sin(\beta_2)\cos(\beta_3)\cos(\beta_4)\sin(\beta_5)\cos(\beta_6)- \\ s_1(t)s_2(t)s_6(t)\sin(\beta_2)\cos(\beta_3)\cos(\beta_4)\cos(\beta_5)\sin(\beta_6) \end{bmatrix} \tag{11}$$

Considering equation (7), equation (11) can be simplified to :

$$\begin{bmatrix} s_1(t)\left[\cos(\beta_2)\cos(\beta_3)\cos(\beta_4)\cos(\beta_5)\cos(\beta_6)-\sin(\beta_2)\sin(\beta_3)\sin(\beta_4)\sin(\beta_5)\cos(\beta_6)\right] \\ +s_1(t)s_5(t)s_6(t)\left[-\cos(\beta_2)\cos(\beta_3)\cos(\beta_4)\sin(\beta_5)\sin(\beta_6)+\sin(\beta_2)\sin(\beta_3)\sin(\beta_4)\cos(\beta_5)\sin(\beta_6)\right] \\ +s_1(t)s_2(t)s_3(t)\left[-\sin(\beta_2)\sin(\beta_3)\cos(\beta_4)\cos(\beta_5)\cos(\beta_6)-\cos(\beta_2)\cos(\beta_3)\sin(\beta_4)\sin(\beta_5)\cos(\beta_6)\right] \\ +s_1(t)s_4(t)s_6(t)\left[-\cos(\beta_2)\cos(\beta_3)\sin(\beta_4)\cos(\beta_5)\sin(\beta_6)+\sin(\beta_2)\sin(\beta_3)\cos(\beta_4)\sin(\beta_5)\sin(\beta_6)\right] \\ +s_1(t)s_3(t)s_4(t)\left[\sin(\beta_2)\cos(\beta_3)\cos(\beta_4)\sin(\beta_5)\cos(\beta_6)-\cos(\beta_2)\sin(\beta_3)\sin(\beta_4)\cos(\beta_5)\cos(\beta_6)\right] \\ +s_1(t)s_2(t)s_6(t)\left[-\sin(\beta_2)\cos(\beta_3)\cos(\beta_4)\cos(\beta_5)\sin(\beta_6)+\cos(\beta_2)\sin(\beta_3)\sin(\beta_4)\sin(\beta_5)\sin(\beta_6)\right] \\ +s_1(t)s_2(t)s_4(t)\left[-\sin(\beta_2)\cos(\beta_3)\sin(\beta_4)\cos(\beta_5)\cos(\beta_6)-\cos(\beta_2)\sin(\beta_3)\cos(\beta_4)\sin(\beta_5)\cos(\beta_6)\right] \\ +s_1(t)s_3(t)s_6(t)\left[-\cos(\beta_2)\sin(\beta_3)\cos(\beta_4)\cos(\beta_5)\sin(\beta_6)+\sin(\beta_2)\cos(\beta_3)\sin(\beta_4)\sin(\beta_5)\sin(\beta_6)\right] \end{bmatrix} \tag{12}$$

with $\beta_2=\beta_3$ and $\beta_4=-\beta_5$, equation (12) yields to:

$$s_1(t)\cos(A+B+C+D+E)=\begin{bmatrix} s_1(t)\cos(\beta_6)\frac{[\cos(2\beta_2)+\cos(2\beta_4)]}{2}+ \\ s_1(t)s_2(t)s_3(t)\cos(\beta_6)\left[\cos^2(\beta_2)\sin^2(\beta_4)-\sin^2(\beta_2)\cos^2(\beta_4)\right]- \\ (s_4(t)-s_5(t))s_1(t)s_6(t)\sin(\beta_6)\frac{\sin(2\beta_4)}{2}- \\ (s_2(t)+s_3(t))s_1(t)s_6(t)\sin(\beta_6)\frac{\sin(2\beta_2)}{2} \end{bmatrix} \tag{13}$$

Considering equations (10) and (13), equation (3) yields to

$$s(t)=\sqrt{2P}\begin{bmatrix} \cos(2\pi f_c t+\phi) & \begin{bmatrix} (s_2(t)+s_3(t))\cos(\beta_6)\frac{\sin(2\beta_2)}{2}+ \\ (s_4(t)-s_5(t))\cos(\beta_6)\frac{\sin(2\beta_4)}{2}+ \\ s_6(t)\sin(\beta_6)\frac{[\cos(2\beta_2)+\cos(2\beta_4)]}{2}+ \\ s_2(t)s_3(t)s_6(t)\sin(\beta_6)\left[\cos^2(\beta_2)\sin^2(\beta_4)-\sin^2(\beta_2)\cos^2(\beta_4)\right] \end{bmatrix} \\ +\sin(2\pi f_c t+\phi) & \begin{bmatrix} s_1(t)\cos(\beta_6)\frac{[\cos(2\beta_2)+\cos(2\beta_4)]}{2}+ \\ s_1(t)s_2(t)s_3(t)\cos(\beta_6)\left[\cos^2(\beta_2)\sin^2(\beta_4)-\sin^2(\beta_2)\cos^2(\beta_4)\right]- \\ (s_4(t)-s_5(t))s_1(t)s_6(t)\sin(\beta_6)\frac{\sin(2\beta_4)}{2}- \\ (s_2(t)+s_3(t))s_1(t)s_6(t)\sin(\beta_6)\frac{\sin(2\beta_2)}{2} \end{bmatrix} \end{bmatrix} \tag{14}$$

## References

[1] E. Rebeyrol. *Galileo Signals and Payload Optimization.* PhD thesis, 2007.