A promising parametric spectral analysis method applied to sea level anomaly signals

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Context of the presentation

SPECTRAL ANALYSIS of sea level anomalies (SLA) widely used in the altimetry community:
To understand the geophysical content of measured signals,
To assess and compare the performance of missions

SPECTRAL ANALYSIS = usually based on Fourier transform

SPECTRUM ESTIMATION

SLA Spectrum

- J2 V0 sla spectrum reference
- J2 V0 sla spec. theoretical
- noise = 6.98
- signal slope = -3.81
- J2 V1 sla spectrum reference
- J2 V1 sla spec. theoretical
- noise = 5.40
- signal slope = -3.62

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NOISE LEVEL

SLOPE
Outline of the talk

1. Spectral analysis based on Fourier transform
2. Spectral analysis based on parametric modeling
3. A parametric spectral analysis for SLA: ARWARP
4. Validation on simulated signals
5. Results on real signals
6. Conclusions and perspectives

Comparisons made in this presentation on simulated Sea Level Anomalies (SLA) and on real signals from SARAL/AltiKa, Agulhas current area
1. Spectral Analysis based on Fourier Transform

Window, \( w(n) \)

SLA signal, \( x(n) \)

Fast algorithm, easy to implement

Fast Fourier Transform

\(|X(f)|^2 \)

\( \hat{S}(f) \)
1. Spectral Analysis based on Fourier Transform

Fast algorithm, easy to implement

SLA signal, $x(n)$

Window, $w(n)$

Fast Fourier Transform

$|X(f)|^2$

$\hat{S}(f)$

Spectral estimation on 1 segment (3000 samples)

SLA AltiKa – Agulhas Current – Cycle 4

Variance
1. Spectral Analysis based on Fourier Transform

- Spectral estimation on 1 segment (3000 samples)
- Averaging several segments (25) (Welch estimation)

<table>
<thead>
<tr>
<th></th>
<th>SLA AltiKa – Agulhas Current – Cycle 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>PSD (m²/ckm)</td>
<td></td>
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</table>

- Fast algorithm, easy to implement
- Window, $w(n)$
- SLA signal, $x(n)$
- Fast Fourier Transform
- $|X(f)|^2$
1. Spectral Analysis based on Fourier Transform

- SLA signal, $x(n)$
- Window, $w(n)$
- Fast algorithm, easy to implement

Fast Fourier Transform $|X(f)|^2 \rightarrow \hat{S}(f)$

Averaging several segments (25) (Welch estimation)

**Blackman-Harris** window

- Variance
- Average spectra
- Variance reduced
- Stationarity?
- Which window?
1. Spectral Analysis based on Fourier Transform

SLA signal, \( x(n) \)

Window, \( w(n) \)

Fast algorithm, easy to implement

Fast Fourier Transform

\[ |X(f)|^2 \]

\( \hat{S}(f) \)

Averaging several segments (25) (Welch estimation)

**Blackman-Harris**

**Tukey**

**Variance**

**Average spectra**

**Variance reduced**

**Stationarity?**

**Which window?**
1. Spectral Analysis based on Fourier Transform

Fast algorithm, easy to implement

SLA signal, $x(n)$

Window, $w(n)$

Fast Fourier Transform

$|X(f)|^2$  $\hat{S}(f)$

Averaging several segments (25) (Welch estimation)

**Blackman-Harris** window

**Tukey** window

Variance

Average spectra

Variance reduced

Stationarity?

Which window?

Different slope estimations

Bias
AutoRegressive (AR) modeling of a SLA signal

\[ x(n) = \sum_{k=1}^{p} a_k x(n - k) + e(n) \]

Actual sample = Linear combination of past samples + unpredictable part (model error)
2. Spectral Analysis based on parametric modeling

AutoRegressive (AR) modeling of a SLA signal

\[ x(n) = \sum_{k=1}^{p} a_k x(n-k) + e(n) \]

Actual sample = Linear combination of past samples + unpredictable part (model error)

Find the AR coefficients \( \{a_1, a_2, \ldots, a_p\} \) such that error is minimized

Several algorithms: e.g., in Matlab lpc.m, levinson.m, aryule.m

2. Spectral Analysis based on parametric modeling

AutoRegressive (AR) modeling of a SLA signal

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Find the AR coefficients \( \{a_1, a_2, \ldots, a_p\} \) such that error is minimized

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2. Spectral Analysis based on parametric modeling

PSD of an audio signal (industrial system)

Welch periodogram in blue
AR spectrum in red (p=30)

- No window choice
- Variance decreased
- Better resolution for short signal segments

What about AR modeling for SLA signals?

- More complex to implement
- Choice of order p
2. Spectral Analysis based on parametric modeling

SLA AltiKa – Agulhas Current – Cycle 4

What you are used to see...

Logarithmic frequency scale
2. Spectral Analysis based on parametric modeling

SLA AltiKa – Agulhas Current – Cycle 4

What you are used to see...

What AR modeling sees...

AR spectral analysis will fit a model on the whole spectrum, on a uniform frequency scale basis.

Necessary to adapt to fit well the interesting part of the PSD.
3. A parametric spectral analysis for SLA: ARWARP

Introduce pre-processing to adapt AR analysis to SLA signals

Frequency warping

Dilatation of the spectrum

Reversible operation

Already used in speech processing

3. A parametric spectral analysis for SLA: ARWARP

\[ f' = f + \frac{1}{\pi} \tan^{-1} \left( \frac{b \sin(2\pi f)}{1 - b \cos(2\pi f)} \right) \]
4. Validation on simulated SLA signals

Simulated SLA, Slope $\alpha = 2.5$, 20 segments of 3000 samples

- Fourier-based PSD
  - Blackman-Harris window
  - Estimated slope $\alpha = 3.25$

- ARWARP PSD
  - Estimated slope $\alpha = 2.29$
4. Validation on simulated SLA signals

Simulated SLA, Slope $\alpha = 2.5$, 20 segments of 3000 samples

- **Fourier-based PSD, Rect. window**
  - mean $\alpha = 2.28$
  - std $\sigma_\alpha = 0.36$
  - MSE = 0.178

- **Fourier-based PSD, B-H. window**
  - mean $\alpha = 2.85$
  - std $\sigma_\alpha = 0.44$
  - MSE = 0.316

- **ARWARP PSD**
  - mean $\alpha = 2.46$
  - std $\sigma_\alpha = 0.29$
  - MSE = 0.086

2000 slope estimations
4. Validation on simulated SLA signals

Simulated SLA, Slope $\alpha = 2.5$, 20 segments of 3000 samples

Fourier-based PSD, Rect. window

- mean $\alpha = 2.28$
- std $\sigma_\alpha = 0.36$
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Fourier-based PSD, B-H. window

- mean $\alpha = 2.85$
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- MSE = 0.316

ARWARP PSD

- mean $\alpha = 2.46$
- std $\sigma_\alpha = 0.29$
- MSE = 0.086

Average both slope estimations?

2000 slope estimations
4. Validation on simulated SLA signals

Simulated SLA, Slope $\alpha = 2.5$, 20 segments of 3000 samples

- **Fourier-based PSD, Rect. window**
  - mean $\alpha = 2.28$
  - std $\sigma_\alpha = 0.36$
  - MSE = 0.178

- **Fourier-based PSD, B-H. window**
  - mean $\alpha = 2.85$
  - std $\sigma_\alpha = 0.44$
  - MSE = 0.316

- **Fourier-based PSD, Rect.+ B-H**
  - mean $\alpha = 2.56$
  - std $\sigma_\alpha = 0.29$
  - MSE = 0.088

- **ARWARP PSD**
  - mean $\alpha = 2.46$
  - std $\sigma_\alpha = 0.29$
  - MSE = 0.086

2000 slope estimations
4. Validation on simulated SLA signals

Simulated SLA, Slope $\alpha = 2.5$, 1 segment of 3000 samples

ARWARP PSD
Estimated slope $\alpha = 2.47$

Fourier-based PSD
Blackman-Harris window
Validation on simulated SLA signals

Simulated SLA, Slope $\alpha = 2.5$, 1 segment of 3000 samples

- Fourier-based PSD, Rect. window
  - $\alpha = 2.73$
  - $\sigma_\alpha = 2.39$
  - MSE = 5.78

- Fourier-based PSD, B-H. window
  - $\alpha = 2.41$
  - $\sigma_\alpha = 1.96$
  - MSE = 3.85

- Fourier-based PSD, Rect.+ B-H
  - $\alpha = 2.57$
  - $\sigma_\alpha = 1.92$
  - MSE = 3.70

- ARWARP PSD
  - $\alpha = 2.15$
  - $\sigma_\alpha = 0.92$
  - MSE = 0.97

2000 slope estimations
5. Results on real signals

SLA SARAL/Altika – cycle 4 – Agulhas Current – **25 segments of 3000 samples**

- Welch, Blackman-Harris, $\alpha = 3.80$
- ARWARP, $\alpha = 3.41$
- Welch, Tuckey, $\alpha = 3.28$
5. Results on real signals

SLA SARAL/Altika – cycle 4 – Agulhas Current – **segment 1 of 3000 samples**

- Welch, Blackman-Harris
- ARWARP, $\alpha = 3.7$
## 6. Conclusions

### Spectral analysis of sea level anomaly signals

<table>
<thead>
<tr>
<th>Fourier-based PSD</th>
<th>ARWARP PSD</th>
</tr>
</thead>
<tbody>
<tr>
<td>For small signal sample size, not interesting</td>
<td>Can be used on small sample size, no need to average</td>
</tr>
<tr>
<td>Necessary to average PSD (Welch)</td>
<td>Averaging PSD possible</td>
</tr>
<tr>
<td><strong>PSD variance</strong></td>
<td>Smooth <strong>PSD</strong></td>
</tr>
<tr>
<td>Estimation of the <strong>slope</strong>: biased (window), large variance</td>
<td>Estimation of the <strong>slope</strong>: small bias, small variance</td>
</tr>
<tr>
<td><strong>When averaging PSDs, slope</strong> estimation combining rectangular and BH windows</td>
<td>( \approx ) ARWARP <strong>slope</strong> estimation (equivalent MSEs)</td>
</tr>
<tr>
<td>Estimation of the <strong>noise level</strong>: good estimator, whatever the window (except rect.)</td>
<td>Estimation of the <strong>noise level</strong>: biased</td>
</tr>
</tbody>
</table>
6. Conclusions

Spectral analysis of sea level anomaly signals using ARWARP or TF-based methods

- Extended paper in preparation: more details, more results
- To be used on other kinds of signals (SLA 1 Hz, wet tropospheric correction, ...)
- Next: study on error bounds
  - Cramer–Rao bound on slope estimation = bound on estimation variance
- Next: study to reduce the slope estimation bias

\[
\min_{a,b} \sum_{k \in J} \left[ \log \hat{S}_x(f_k) - a f_k - b \right]^2
\]

Slope estimation = Linear regression

\[
\min_{\theta} \sum_{k \in I} \left[ \log \hat{S}_x(f_k) - \log S_x(f_k; \theta) \right]^2
\]

Slope estimation = Model fitting
Thank you for your attention