Stable probability laws modeling random propagation times of waves crossing different media

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Abstract—In a communication scheme, there exist points at the transmitter and at the receiver where the wave is reduced to a finite set of functions of time which describe amplitudes and phases. For instance, the information is summarized in electrical cables which preceed or follow antennas. In many cases, a random propagation time is sufficient to explain changes induced by the medium. In this paper we study models based on stable probability laws which explain power spectra due to propagation of different kinds of waves in different media, for instance, acoustics in quiet or turbulent atmosphere, ultrasonics in liquids or tissues, or electromagnetic waves in free space or in cables. Physical examples show that a sub-class of probability laws appears in accordance with the causality property of linear filters.

keywords: random propagation time, stationary process, stable probability law, causality.

I. INTRODUCTION

Acoustic waves propagate through shocks between molecules which move in the medium. Consequently any (equivalent) time of propagation has a random nature. The same property is true for electromagnetic waves which propagate in a medium which is never totally empty and then which is submitted to fluctuent fields. In both cases the result $\mathbf{U} = \{U(t), t \in \mathbb{R}\}$ of propagation of the wave $\mathbf{Z} = \{Z(t), t \in \mathbb{R}\}$ can be modelled as

$$U(t) = Z(t - A(t)) \tag{1}$$

where the random process $\mathbf{A} = \{A(t), t \in \mathbb{R}\}$ is an equivalent propagation time.

This paper addresses stable probability laws modelling propagation times of waves [1], [2], [3], [4]. The main effects of propagation are weakenings, changes in phases and amplitudes, widenings of power spectra and Doppler effects. I have shown that random propagation times explained these transformations. For instance, in acoustics and ultrasonics, where changes are ruled by the Beer-Lambert law, the (nearly) Kramers-Krönig relations and selective attenuations [5], [6], [7]. Also in electromagnetics, where literature highlights spectral widenings and/or Doppler shifts. Examples can be taken in radar domain [8], [9] or in optics for laser or natural light [10], [11]. Propagation through coaxial cables shows a particular form of weakenings and phase changes [12]. Let consider an acoustic wave propagating in atmosphere or water. The path can be divided into a sum of (approximately) independent pieces. The application of some version of the central limit theorem leads to a Gaussian probability law for the random propagation time $\mathbf{A}$, the random behavior being due to small and random changes of the refraction index. We show in section 3 that this explains the weakening of pure tones in $\exp\left[-a\omega^2\right]$ observed most of the time in acoustics ($\omega/2\pi$ is the frequency), and many kinds of spectra [13]. But ultrasonics show more general attenuations in $\exp\left[-a\omega^b\right], 0 < b \leq 2$, where b is often different of 2. Stable probability laws generalize the central limit theorem and can be involved in this situation [1], [2], [3]. They can be asymmetric when $\alpha \neq 2$, and we will see that this property allows to explain wave dispersion (the celerity depends on the frequency).

The next section studies properties due to stationary random propagation times. The Gaussian case and the more general case of stable laws are explained in sections 3 and 4. Applications will be given in section 5 when stable probability laws may be appealed. It is the case for acoustics, ultrasonics and propagation through coaxial cables or in power cables, where experiments can be done for wide enough frequency bands. These examples highlight a particular sub-class of probability laws which is in accordance with the causality property of linear filters. In the appendix, we explain the construction of stationary stable processes which seem well matched to given examples.

II. RANDOM PROPAGATION TIMES

1) In what follows, we assume that $\mathbf{Z}$ (the real or complex random process which propagates) is zero-mean and wide sense stationary with power spectral density $s_Z(\omega)$ defined by

$$\mathrm{E}\left[Z(t)Z^*(t-\tau)\right] = \int_{-\infty}^{\infty} e^{i\omega\tau} s_Z(\omega)\, d\omega. \tag{2}$$

$\mathrm{E}[..]$ stands for the mathematical expectation (or ensemble mean), the superscript * for the complex conjugate and $s_Z(\omega)$ can be a mixing of regular functions and "Dirac functions". $\mathbf{A}$ (the random propagation time involved in (1)) is a random process independent of $\mathbf{Z}$ and such that the characteristic functions

$$\begin{cases} \mathrm{E}\left[e^{-i\omega A(t)}\right] = \psi(\omega) \\ \mathrm{E}\left[e^{-i\omega(A(t)-A(t-\tau))}\right] = \phi(\tau, \omega) \end{cases} \tag{3}$$

are independent of t. This defines a stationarity stronger than the usual wide sense stationarity. The result of propagation $\mathbf{U}$ defined by (1) can be split in the orthogonal sum (see [14] and the appendix)

$$\begin{aligned} U(t) &= G(t) + V(t) \\ \mathrm{E}\left[G(t)V^*(t-\tau)\right] &= 0 \end{aligned} \tag{4}$$

whatever $\tau \in \mathbb{R}$. The random process $\mathbf{G}$ is the output of a linear invariant filter with input $\mathbf{Z}$ and complex gain $\psi(\omega)$. This means that

$$\begin{cases} G(t) = \int_{-\infty}^{\infty} Z(t-u) f(u)\, du \\ \psi(\omega) = \int_{-\infty}^{\infty} e^{-i\omega u} f(u)\, du \\ s_G(\omega) = \left[|\psi|^2 s_Z \right](\omega) \end{cases} \quad (5)$$

for regular $f(u)$ (it is together the probability density of $-A(t)$ and the filter impulse response). $s_G(\omega)$ is the power spectral density of $\mathbf{G}$. Moreover

$$K_V(\tau) = \mathrm{E}\left[V(t) V^*(t-\tau) \right] = \int_{-\infty}^{\infty} \left[\phi(\tau, \omega) - |\psi(\omega)|^2 \right] e^{i\omega\tau} s_Z(\omega)\, d\omega. \quad (6)$$

We may assume that

$$\lim_{\tau \to \infty} \phi(\tau, \omega) = |\psi(\omega)|^2$$

with a fast enough convergence. This means that $A(t)$ and $A(t-\tau)$ become independent when τ is large enough. In this case, $\mathbf{V}$ will have a regular power spectral density $s_V(\omega)$ defined by

$$K_V(\tau) = \int_{-\infty}^{\infty} e^{i\omega\tau} s_V(\omega)\, d\omega. \quad (7)$$

2) Let assume that devices transmit a pure spectral line at the frequency $\omega_0/2\pi$

$$Z(t) = e^{i\omega_0 t}.$$

The fact that the mean depends on t has no importance. The received wave verifies, from $(5), (6)$

$$\begin{cases} G(t) = e^{i\omega_0 t} \psi(\omega_0) \\ 2\pi s_V(\omega) = \\ \int_{-\infty}^{\infty} \left[\phi(\tau, \omega_0) - |\psi(\omega_0)|^2 \right] e^{i(\omega_0 - \omega)\tau} d\tau. \end{cases} \quad (8)$$

The result of the propagation is the sum of a pure wave $\mathbf{G}$ (which ends to a DC part after an amplitude demodulation of shift ω_0) and a band process $\mathbf{V}$ (the AC part). The powers of the components are equal to $|\psi(\omega_0)|^2$ (for the DC one) and

$$K_V(0) = 1 - |\psi(\omega_0)|^2$$

(for the AC one). The sum of powers is equal to 1, i.e. the propagation is lossless.

The DC/AC ratio r is the ratio of the respective powers [15]

$$r = \frac{|\psi(\omega_0)|^2}{1 - |\psi(\omega_0)|^2}. \quad (9)$$

Actually, devices of reception take into account a limited band around the frequency ω_0. As a consequence, it happens that only the monochromatic part is viewed in more or less important noise which is the sum of a surrounding medium noise and a little part of the companion process $\mathbf{V}$ defined by $(4), (5)$ and (6). It is often the case in acoustics [5] and in ultrasonics [7] or in propagation in coaxial cables [12]. Conversely the spectral line can disappear. For instance for electromagnetic waves in free space. Light coming from stars shows lines of hydrogen H_α, H_β... and of other elements with various widths [11]. We know that laser emissions are broadened when propagating in space [16]. Moreover, in the computation of power spectra, the autocorrelation function is cancelled for missing data (apodization). This leads to a widening of pure spectral lines which can disappear in the surrounding noise [13].

III. THE GAUSSIAN CASE

1) Let assume that $\mathbf{A}$ is a Gaussian process. If the propagation time can be considered as a sum of approximately independent components (for instance successive thicknesses in the atmosphere), its mean, its variance and its covariance are proportional to the propagation distance l. We assume that these quantities depend only on the crossed medium (and not on the wave properties). Then, the characteristic functions of (3) are defined by

$$\begin{cases} \psi(\omega) = \exp\left[-iml\omega - l(\sigma\omega)^2/2 \right] \\ \phi(\tau, \omega) = \exp\left[-l(\sigma\omega)^2 (1 - \rho(\tau)) \right] \\ l\sigma^2 \rho(\tau) = \mathrm{E}\left[A(t) A(t-\tau) \right] \end{cases} \quad (10)$$

where m is the mean, σ the standard deviation and $\rho(\tau)$ the (normalized) autocorrelation function for the unit distance. From (8), we have

$$G(t) = e^{i\omega_0(t-ml) - l(\sigma\omega_0)^2/2}. \quad (11)$$

ml is the transit time of the pure wave on the distance l. The celerity of the wave $\mathbf{G}$ is equal to m^{-1} and does not depend on the frequency. Equivalently the refraction index is constant (the medium is not dispersive). The dependence on l of the amplitude and the phase of $G(t)$ is a proof that we are in the Beer-Lambert law context. Beside the monochromatic wave $\mathbf{G}$ appears the process $\mathbf{V}$ with power spectral density defined by (from (8))

$$2\pi s_V(\omega) = \int_{-\infty}^{\infty} e^{-i(\omega - \omega_0)\tau - l(\sigma\omega_0)^2} \left[e^{l(\sigma\omega_0)^2 \rho(\tau)} - 1 \right] d\tau. \quad (12)$$

It is possible to choose $\rho(\tau)$ in sort that $s_V(\omega)$ takes very small values in the frequency bands viewed by devices. Then, $\mathbf{V}$ will be neglected and/or plugged in other surrounding noises.

2) Let assume that the behavior of $\rho(\tau)$ at the origin point verifies

$$\rho(\tau) = 1 - \left| \frac{\tau}{\tau_0} \right|^b + o\left(\left| \frac{\tau}{\tau_0} \right|^b \right) \quad (13)$$

with $0 < b \leq 2$. (13) rules the behavior of $\rho(\tau)$ nearby the point $\tau = 0$. $\rho(\tau)$ takes into account the celerity of variations of the propagation time $\mathbf{A}$ by length unit. Practically, from (12), this means that we have

$$s_V\left(\omega_0 + \omega \frac{\mu^{1/b}}{\tau_0} \right) \cong \frac{\tau_0 \mu^{-1/b}}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega u - |u|^b} du \quad (14)$$
$$\mu = l(\omega_0 \sigma)^2$$

when μ is sufficiently large. In the same time, the bandwidth of $\mathbf{V}$ increases when τ_0 decreases. τ_0 has the same meaning as a "decorrelation time".

3) To a better understanding, let assume that $\omega_0/2\pi = 10^6$ (for instance in ultrasonics), $l = 1$ and

$$\omega_0\sigma = 1, \rho(\tau) = \begin{cases} 1 - |\tau/\tau_0|, \left|\frac{\tau}{\tau_0}\right| < 1 \\ 0, |\tau/\tau_0| \geq 1 \\ \tau_0 = 10^{-9}. \end{cases}$$

We have respectively $P_G = e^{-1}$ (it is the power of **G,** the height of the residual spectral line**)**, $P_G + P_V = 1$ (P_V is the power of **V).** $s_V(\omega)$ is almost constant in the interval $(\omega_0 - 10^8, \omega_0 + 10^8)$ with value less than 10^{-9}. Clearly, the process **V** cannot be distinguished by devices matched to the extraction of the line **G** (i.e centered around the frequency $\omega_0/2\pi$**).**

Conversely, when $\omega_0/2\pi = 10^{14}, l = 1$ (for instance in infrared), let assume that

$$\omega_0\sigma = 4, \rho(\tau) = \begin{cases} 1 - |\tau/\tau_0|, |\tau| < \tau_0 \\ 0, |\tau| \geq \tau_0. \end{cases}$$

We obtain $P_G \simeq 0, P_V \simeq 1$. Furthermore, provided that τ_0 is not too large

$$\pi s_V(\omega + \omega_0) \simeq \frac{\theta}{\theta^2 + \omega^2}, \theta = \frac{16}{\tau_0}.$$

ω_0 is the gravity center of $s_V(\omega)$, and $4/\sqrt{\tau_0}$ is a measure of the bandwidth. If $l = 2$ and not $l = 1$, we obtain $\theta = 32/\tau_0$. This means that increases of l (like decreases of τ_0) lead to a flattening (a widening) of $s_V(\omega)$. The Lorentzian behavior of $s_V(\omega)$ is related to the value $b = 1$ in (13). (14) defines the links between the value of b and the shape of $s_V(\omega)$. For instance $b = 2$ leads to a Gaussian spectrum.

IV. STABLE PROBABILITY LAWS

1) The probability law of $A(t)$ is stable when the characteristic function (c.f) $\psi(\omega) = \mathrm{E}\left[e^{-i\omega A(t)}\right]$ defined in (3) has the shape [1], [2], [3]

$$\ln \psi(\omega) = \\ -im\omega - a|\omega|^\alpha \left(1 + i\beta\frac{\omega}{|\omega|}h(|\omega|, \alpha)\right) \quad (15)$$

with real parameters $(-m, a, \alpha, \beta)$ and $a > 0, 0 < \alpha \leq 2, |\beta| \leq 1$. Moreover

$$h(|\omega|, \alpha) = \begin{cases} \tan \frac{\pi\alpha}{2}, \alpha \neq 1 \\ \frac{2}{\pi}\ln|\omega|, \alpha = 1. \end{cases} \quad (16)$$

The corresponding random variables have probability densities with analytic closed form in three cases, $\alpha = 2$ (Gaussian), $\alpha = 1, \beta = 0$ (Cauchy/ Lorentz) and $\alpha = 1/2, \beta = \pm 1$ (Levy). In other cases, series developments can be used [2], [3]. The laws are always unimodal and they are one-sided only when $0 < \alpha < 1, \beta = \pm 1$. Formula (15) is matched to an unit propagation length. For a length l, we can replace A by A_l, m by ml and a by al. This means that the r.v $A_l(t)$, the propagation time on a length l is the sum of l r.v which have the same probability law that $A(t)$ (for any integer l).

In the notations of [2], $\psi(\omega)$ is the c.f of the r.v $-A(t)$. Then, the opposite $A(t)$ follows a stable law of parameters $(m, a, \alpha, -\beta)$. We will see that the value of β has a great

importance. When $\beta = 1, 0 < \alpha < 1, m = 0$, it is proved that the r.v $A(t)$ is such that

$$\Pr[A(t) > 0] = 1.$$

Equivalently, the filter of complex gain $\psi(\omega)$ is causal, because the impulse response of this filter is the probability density of $A(t)$ (look at (3) and (6)).

For $1 < \alpha < 2$, the r.v $A(t)$ is never one-sided. Nevertheless, the value $\beta = 1$ provides the most asymmetric possible probability density, so that this value is the best choice with respect to the causality property. When $\alpha = 1$, we have to take the value $\beta = -1$, which gives the same property. For $\alpha = 2$, the value of β is indifferent.

2) The choice of this class of probability laws will be justified in the following section by practical examples. Experiments often show that the r.v. $A(t)$ has to be Gaussian for each t (the particular case $\alpha = 2$). In this situation it is natural to assume that the entire process **A** is Gaussian which implies that the margins $A(t)$ and the linear combinations $A(t) - A(t - \tau)$ are Gaussian. It is not a great bet though the knowledge of margins does not define entirely the underlying probability laws most of the time. This kind of problem is complex [18].

The propagation of ultrasonic waves involves stable laws more general than the Gaussian one, see section 5.2 and [6], [7]. In appendix, we prove that it is possible to construct processes **A** such that both $A(t)$ and $A(t) - A(t - \tau)$ follow stable laws with the same exponent α, whatever its value in $]0, 2]$. This means that the Gaussian model can be naturally generalized. To take α different of 2 in (15) does not really limit the possibilities for spectral densities of **V** (defined in section 2). In this expanded context it is possible to model propagations with mixed results (neither **G** nor **V** is negligible [17]).

3) We have seen in section 2 that $\psi(\omega)$ can be viewed as the complex gain of a linear invariant filter (which provides the process **G**). $\psi(\omega_0) \exp[i\omega_0 t]$ is the answer of some device to the pure tone $\exp[i\omega_0 t]$. In the Gaussian framework which is described by (10), the wave amplitude is weakened by $\exp\left[-l\omega_0^2\sigma^2/2\right]$ and delayed by the constant ml. $c = 1/m$ is the celerity of the wave in the medium and does not depend on ω_0. More generally, when $\psi(\omega)$ verifies (15), the wave weakening is equal to

$$\exp[-a|\omega_0|^\alpha].$$

The propagation time of the monochromatic part **G** is no longer a constant (apart when $\alpha = 2$) but it becomes the quantity $(\omega_0 > 0)$

$$m + a\beta\omega_0^{\alpha-1}\begin{cases} \tan(\pi\alpha/2), \alpha \neq 1 \\ \frac{2}{\pi}\ln\omega_0, \alpha = 1. \end{cases}$$

Except for $\alpha = 2$ (the Gaussian case), this means that we deal with a dispersive medium in which the celerity is a function of the frequency. Experiments described below prove that the parameter β in (15) has to take one of particular values 1 or -1. It is of huge importance for the model validity because these values are closely linked to the notion of causality (i.e the consequence of an action appears only after the action).

4) Figures 1, 2, 3 illustrate stable laws properties linked to probability densities (which are always regular and unimodal). Figure 1 depicts the case $(0,1,1/2,\beta)$, for some values of β. The value $\beta = -1$ provides the only one-sided probability density (which leads to causality). In Figure 2, the case $(0,1,3/2,\beta)$ is studied. The value $\beta = -1$ does not provide a one-sided probability density, but it is the best bet for approaching the causality property. Figure 3 shows the case $(0,a,3/2,-1)$. We see that the "width" of the probability density decreases in the same time that a.
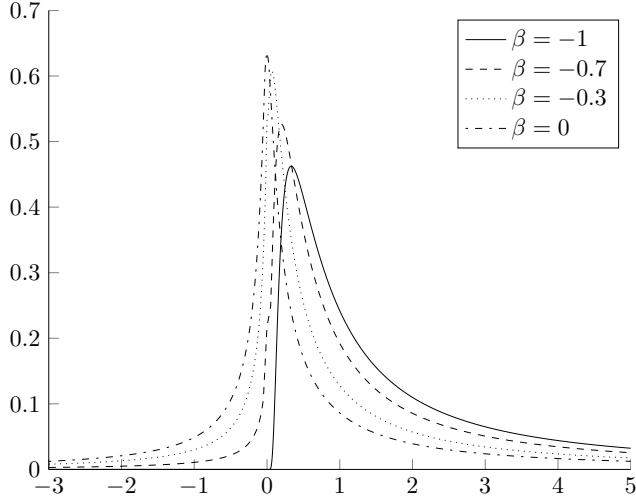

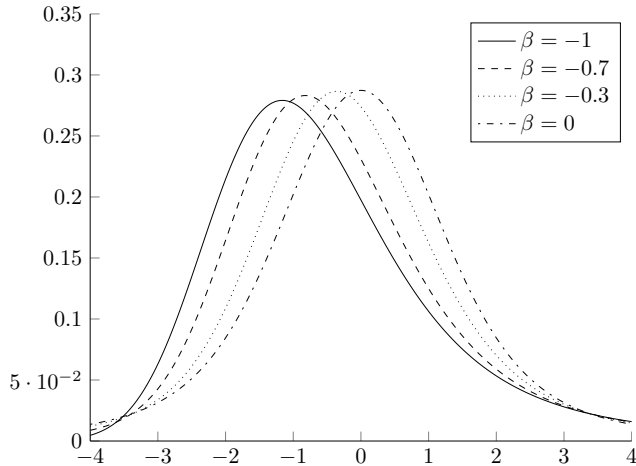Fig. 1. Probability density of $A(t)$ for $\alpha = \frac{1}{2}$, $a = 1$ as a function of β

Fig. 2. Probability density of $A(t)$ for $\alpha = \frac{3}{2}$, $a = 1$ as a function of β

V. EXAMPLES

A. Acoustics

No propagation phenomenon was better studied than acoustic waves through atmosphere and sea water (mainly for military aims in the latter case). In frequency bands where physical conditions in the medium are invariant, the complex amplitude $F_l(\omega)$ of the monochromatic wave $e^{i\omega t}$ measured
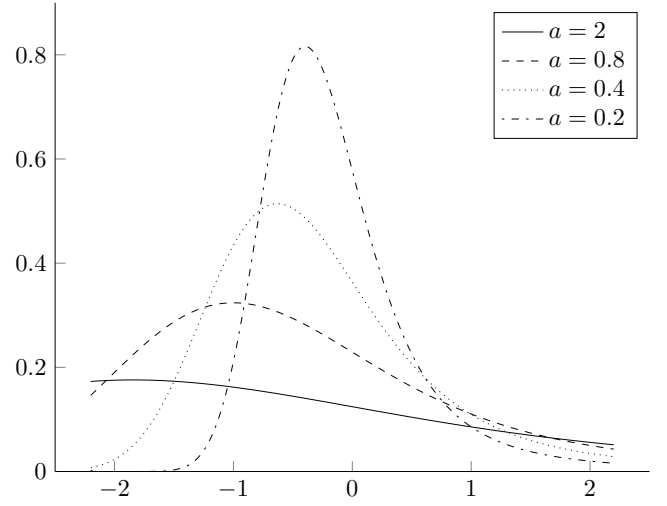

Fig. 3. Probability density of $A(t)$ for $\alpha = \frac{3}{2}$, $\beta = -1$ as a function of a

at the distance l is in the form ($\omega/2\pi$ is the frequency)

$$F_l(\omega) = e^{-l[a\omega^2 + i\omega/c]} \tag{17}$$

where a depends on the temperature, viscosity, composition.... The celerity c of the wave has the same property [19], [20]. (17) is true for frequencies going up to dozens of MHz and whatever the path length l up to kilometres if the medium is subjected to constant conditions. These conditions are able to change because some kinds of molecules in atmosphere (H_2O, CO_2..) or in water ($MgSO_4$, $B(OH)_3$..) interact at characteristic frequencies. a is constant in frequency bands which depend on the kind of molecules and on their concentrations. In frequency bands where a and c can be considered as constant, (17) shows that the medium on a length l reacts like a linear invariant filter of complex gain $F_l(\omega)$. Consequently, the Gaussian model of propagation studied in section 3 is available. The random propagation time **A** is a Gaussian process with $\psi(\omega) = F_l(\omega)$. The received wave is the component **G.** The component **V** has likely a flat power spectrum which makes it invisible among ambient noises, because measurements devices are matched to the transmitted frequency. Let note that literature is very detailed for variations of a with respect to frequency, temperature and other parameters. But I do not know references about variations of c versus the frequency. This means that we have to admit that the medium is not dispersive in this occurence.

B. Ultrasonics

Ultrasonic waves crossing various materials on small distances (for instance for echography) lead to more general formulas [7]. In this case, we often measure complex amplitudes $F_l(\omega)$ (for monochromatic transmitted waves at the frequency $\omega/2\pi$ on a length l)

$$F_l(\omega) = e^{-i\omega(l/c) - al|\omega|^\alpha (1 + i(\mathrm{sgn}\omega)\tan(\pi\alpha/2))}$$
$$0 < \alpha \leq 2, \alpha \neq 1. \tag{18}$$

or, for $\alpha = 1$

$$F_l(\omega) = e^{-i\omega(l/c) - al|\omega|\left(1 - \frac{2}{\pi}i(\mathrm{sgn}\omega)\ln|\omega|\right)}$$

We recognize particular cases of (15) about stable probability laws with $\beta = \pm 1$.

This model is available with $\beta = 1$ for very different media like castor oil ($\alpha = 1.66$), polyethylene ($\alpha = 1.1$), rubber ($\alpha = 1.38$) and also brass tube ($\alpha = 0.5$ but for acoustic low frequencies) [21], [22], [23]. The case $\alpha = 1$ (with $\beta = -1$) is encountered for instance when modeling evaporated milk. The value $\beta = \pm 1$ is often well-fitted and has a particularly physical meaning. When $\alpha < 1$, the probability density of $F_l(\omega)$ is one-sided (it is false for other values of β). Equivalently, the linear invariant filter of complex gain $F_l(\omega)$ is causal (the result $\mathbf{G}$ is viewed by devices). When $\alpha \geq 1$, these values of β minimize a tail of distribution or equivalently they are the most favourable with respect to the causality property. Of course it is a huge argument for physicists. Values of β different of ± 1 are also encountered for example for egg yolk ($\alpha = 1.54, \beta = 0.85$) and lucite ($\alpha = 0.625, \beta = 0.71$) [7]. But measurements of the celerity in the medium which rules the value of β are difficult and can be less accurate than measurements of α which rules the attenuation variations. Like in the acoustic case, the second component (noted $\mathbf{V}$ in section 2) is not detected by devices. This means that the $\mathbf{V}$-power spectrum is very flat and spread with respect to the frequencial window of reception devices.

C. Coaxial cables

Propagation in coaxial cables can be compared to propagation of acoustic or ultrasonic waves (Drude model). In the latter cases, the propagation is due to interactions between molecules and in the first case, to interactions between electrons of conduction and the medium. From a classical theory and for an infinite line, a length l of cable is equivalent to a linear invariant filter of complex gain

$$F_l(\omega) = e^{-l\sqrt{(R+iL\omega)(G+iC\omega)}}$$

where R, L, C, G are the series resistance, the series inductance, the shunt capacitance and the shunt conductance by length unit. Actually, R depends on ω due to the "skin effect", G is small, which leads to the formula

$$F_l(\omega) = e^{-l\left(im\omega + a\sqrt{|\omega|}(1 + i\mathrm{sgn}\omega)\right)}. \qquad (19)$$

This formula is for example available for the Belden 8281 cable up to 1GHz [24], [12]. Then we see that $-A(t)$ follows a stable probability law of parameters (see section 4)

$$(-ml, al, 1/2, 1).$$

The parameter $1/m$ (the wave celerity) is close to 2/3 the light celerity in the vacuum, a depends on the manufacturing process. The value $\alpha = 1/2$ is not encountered in ultrasonics (at my knowledge we have $\alpha \geq 1$ in this domain) but we find this value in [23] (propagation in brass tubes at low frequency). We have explained in the section above that the value $\beta = 1$ has a particular sense. It means that the propagation on a distance l is equivalent to a causal linear invariant filter with complex gain $F_l(\omega)$. The causality property is verified only for this value of β. As usual, the output of this filter is identified to the process $\mathbf{G}$ defined in section 2. The process $\mathbf{V}$ summarizes losses which are not taken into account by devices.

The stable probability law model is available for other conductors like XLPE cables for energy transmission. In figure 1 of [25], attenuation versus frequency is well-fitted by $y = ax^\alpha$ (in dB) with $\alpha = 1.13$ up to 50MHz for a cable in a good state of repair. Unfortunately, the authors do not give curves about the celerity, and it is not possible to verify the shape of the imaginary part in (15) (and then the value of the parameter β). Figure 4 in this paper shows the fit of attenuation for data in [26] and three species of cables in good state. We find $\alpha = 1.23 - 1.24$. Moreover the study of data giving the variation of the wave celerity leads to the values

$$\beta = 1 - 0.74 - 0.66.$$

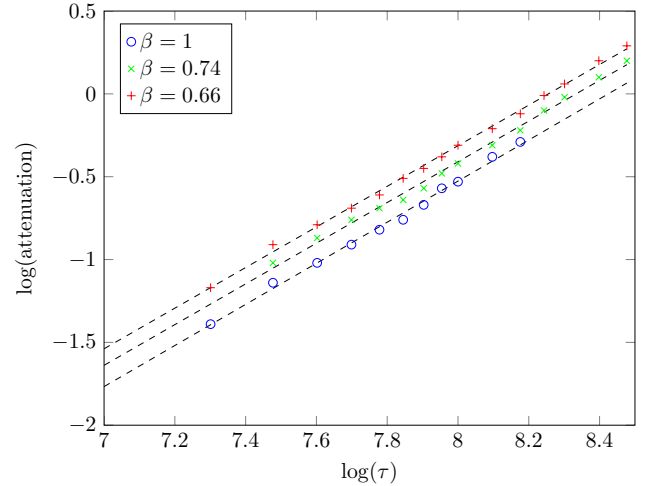Recall that the causality of filters corresponds to the value $\beta = 1$.

Fig. 4. Log(attenuation) plot of XLPE cable from measurements in [26]

D. The Mintzer's law

Propagation of acoustics in water or atmosphere is well explained by Gaussian propagation times (see section 5.1). The parameters are defined by the macroscospic characteristics of the medium, like temperature, viscosity, composition.... About the propagation of the monochromatic wave $e^{i\omega_0 t}$ on the distance l, the amplitude is constant and equal to $e^{-al\omega_0^2}$ (see section 5.1). In the case of a turbulent medium, for instance induced by a heated grid [27], we may consider that the weakening parameter a is a slowly time-variant quantity $B(t)$, which has a random behavior linked to the strength of turbulences. Consequently, the wave amplitude $C(t)$ at the distance l becomes the random quantity

$$C(t) = \exp\left[-B(t)l\omega_0^2\right]. \qquad (20)$$

Classically, the "coefficient of variation" μ of the r.v (random variable) $C(t)$ is defined by [15]

$$\mu = \frac{\sqrt{\mathrm{Var}C(t)}}{\mathrm{E}\left[C(t)\right]}.$$

The Mintzer's law asserts that μ varies like $\omega_0 \sqrt{l}$ [28]. We ask to find probability laws for $B(t)$ which are able to explain such a result (we assume that the process $\mathbf{B}$ is stationary).

Let assume the existence of the moment-generating function

$$\xi(s) = \mathrm{E}\left[e^{-sB(t)}\right], s \geq 0.$$

It is a strong hypothesis when the probability laws are not one-sided [3]. We have the equality

$$\mu = \frac{\sqrt{\xi\left(2l\omega_0^2\right) - \xi^2\left(l\omega_0^2\right)}}{\xi\left(l\omega_0^2\right)} \quad (21)$$

When $B(t)$ follows a stable law with parameters $(m', \theta, 1, 1)$, we know that $\xi(s)$ exists and that [3]

$$\xi(s) = \exp\left[\theta s \ln s - m's\right]. \quad (22)$$

In this case, (21) becomes

$$\mu = \sqrt{\exp\left[2\theta l\omega_0^2 \ln 2\right] - 1}.$$

For small μ corresponding to small $l\omega_0^2$, we have

$$\mu \sim \sqrt{2\theta \ln 2}\,\omega_0 \sqrt{l} \quad (23)$$

in agreement with the Mintzer's law. Experimental values found in literature (see [5]) justify the limited development. For stable laws with parameters $(m, \theta, \alpha, -1), \alpha \neq 1$, $\xi(s)$ exists (and not in other cases, except the preceeding one [3]), and we obtain

$$\mu \sim \sqrt{\theta\left|2 - 2^\alpha\right|}\left(l\omega_0^2\right)^{\alpha/2}$$

which returns to (23) when $\alpha \to 1$. In [5], we show that the half-Cauchy law, which is not stable, leads to the Mintzer's property. It is a law with discontinuous probability density, but which has the advantage to be one-sided (it is not true for (22)). Let note that physicists study moments of the log-amplitude. In the situations which are developed in this section, these quantities do not exist.

E. Backscatter from trees

In paper [29], a radar at 8GHz interacts with crowns of trees. Each experiment addresses a particular tree among six species, and at two different wind celerities. Measurements of autocorrelation functions $K(\tau)$ of the backscatter are compared with the so-called Wong model (a mixture of Gaussians) [31]. If we consider the curves up to the "decorrelation time" τ_0 (defined by $K(\tau_0) = 1/e$), they are very well fitted by (after demodulation in baseband)

$$K(\tau) = \exp\left[-(\tau/\tau_0)^\alpha\right]$$

where $1.65 \leq \alpha \leq 2$, following the specy of tree and the wind speed [8]. We know that the accuracy of measurements decreases with τ, due to apodization and variations of the frequency. It is the reason which leads to neglect measurements above the decorrelation time, and in the same time to admit that $\lim_{\tau \to \infty} K(\tau) = 0$. Nevertheless, all data of [29] drawn in the set of coordinates

$$(x = \ln \tau, y = -\ln\left[-\ln K(\tau)\right])$$

are above the line

$$y = \alpha x - \alpha \ln \tau_0$$

which cannot be due only to inaccuracies. Taking into account data above τ_0 leads to the model

$$K(\tau) = c + (1 - c)\exp\left[-(\tau/\tau_0)^\alpha\right].$$

We see that a specular component appears with a power equal to c and a DC/AC ratio r equal to (see(9))

$$r = c/(1 - c).$$

Figures 5 and 6 from [13] illustrate the fits for apple tree and white mulberry. Gaussian propagation times explain these results. Neither the $\mathbf{G}$ part (here the DC component) nor the $\mathbf{V}$ part (the AC component) disappears.
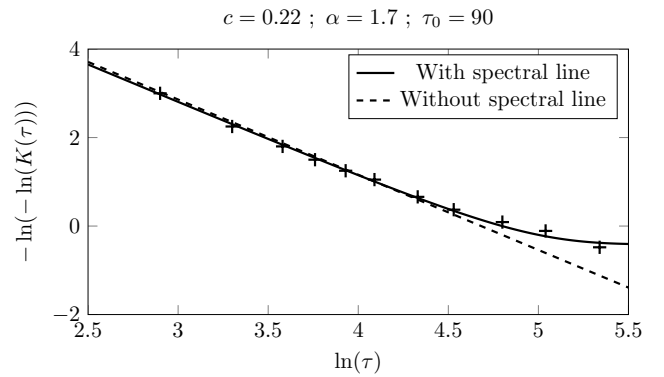

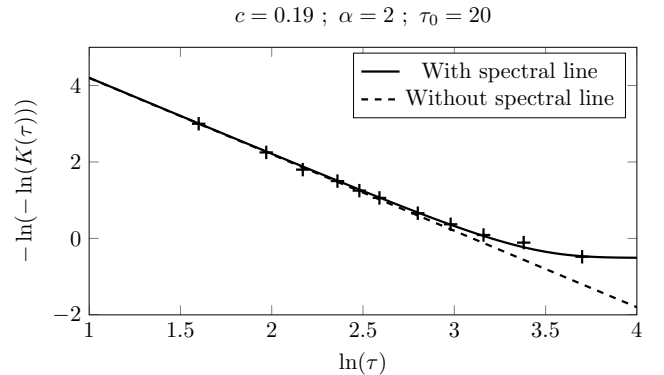Fig. 5. Autocorrelation plot of apple tree at a windspeed of 3.8 m/s

Fig. 6. Autocorrelation plot of white mulberry at a windspeed of 7.9 m/s

F. Electromagnetic waves in free space

Generally, propagation of electromagnetic waves in free space or atmosphere leads to weakenings, spectral widenings and Doppler shifts. The disappearance of spectral lines is easily explained by Gaussian random propagation times. The weakening of the line $\exp\left[i\omega_0 t\right]$ is measured by the quantity $\exp\left[-\omega_0^2\sigma^2/2\right]$ where the standard deviation $\sigma = \sqrt{\mathrm{Var}\,A(t)}$ characterizes the amplitude of time variations of the wave trajectory. For a He/Ne laser at $0.633\mu m$ (in the visible) on a propagation distance of 15m, $\omega_0\sigma = 10$ for $\sigma = 3.10^{-14}$s.

This value corresponds to 10^{-6} times the mean propagation time, which is likely much smaller than uncertainties about the propagation length. Clearly, the line disappears and is totally replaced by a continuous spectral component. In this situation, the characteristic function $\psi(\omega)$ cannot be measured. Even when a part of the line can be highlighted [13], the transmitters (radar, laser,...) work generally at only one frequency, which does not allow to measure functions of the frequency. Nevertheless, the classical central limit theorem works for acoustics and ultrasonics in free space (ω_0 has not the same order of magnitude, it remains a pure spectral line, see sections 5.1 and 5.2), and, because the lack of data, the Gaussian model seems the best bet about electromagnetic propagation in free space. In this context, the behavior close to the origin of $\rho(\tau)$ defined in (13), allows to fit spectral densities in the form

$$K_V(\tau) = a \exp\left[-b\tau^\alpha\right]$$

where $0 < \alpha \leq 2$ [13]. Moreover, Doppler shifts appear when moving targets and /or continuous spectra of transmitters [8], [11], [9].

VI. Remarks

1) We can admit that, when the model of stable laws is available, the value of β is 1 or -1 following that α is equal or different from 1 (see examples 2 and 3). These values correspond to the best value with respect to the causality property (strictly verified only when $0 < \alpha < 1$). Consequently, when the propagation is free of dispersion (the wave celerity is constant whatever the frequency), the only possibility is $\alpha = 2$ (the Gaussian case). It seems that this situation appears for propagation in free space, together for acoustics and for electromagnetic waves, though physical models are very different.

2) In literature, the Kramers-Krönig relations were proved under the causality condition. If the (integrable) impulse response $f(t)$ of a filter verifies

$$f(t) = 0, t < 0$$

and if its Fourier transform (the complex gain)

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$$

is such that $|F(\omega)| \rightarrow_{|\omega| \to \infty} 0$ faster than $1/|\omega|$, the real and imaginary parts of $F(\omega)$ are Hilbert transforms the one with respect to the other (except for a sign). The proof is based on the fact that $F(z)$ is analytic in the lower half plane. Equivalently, the Laplace transform $F(z)$ of $f(t)$ exists for $y < 0$ ($z = x + iy$). In this paper, the characteristic function $\psi(\omega)$ is the complex gain. We know that, for the admitted values of β (1 or -1 following the α value), $\psi(z)$ has the property of analicity and its behavior at $\pm\infty$ is fast enough [3]. Consequently, the Kramers-Krönig relations are true, even if the probability density of $\psi(\omega)$ does not cancel, which is the case for $1 \leq \alpha < 2$. In this case, the causality condition is not verified.

3) We have admitted in many cases that $A(t)$ follows a stable law with parameters $(l/c, al, \alpha, \pm 1)$. l/c is a parameter of position and al is a scale parameter. The dependency in l is a consequence of the Beer-Lambert law. When $1 < \alpha \leq 2$, the filter linked to $\psi(\omega)$ is not causal, but this drawback becomes more and more weak when l increases, because $A(t)/l$ follows a stable law with parameters

$$\left(1/c, al^{1-\alpha}, \alpha, \pm 1\right)$$

which shows that the ratio scale parameter versus parameter of position is more and more weak. For $\alpha = 1$, the ratio is constant.

VII. Conclusion

I had shown for a long time and in this journal that random clock changes provide interesting models for wave propagation [14], [31], [32]. Gaussian processes soon gave an interesting panel of applications, but they often are not available outside the atmosphere.

The examples given in section 5 show that random propagation times following Gaussian or non-Gaussian stable probability laws are good models in acoustics, ultrasonics and in the domain of electromagnetic waves. Except for the Gaussian, the used laws are not symmetric and the most possible asymmetric (symmetric stable laws are referred as SαS in literature and widely used [30]). The extreme case of asymmetry appears for the values $\beta = \pm 1$. This occurence is found in the given examples and it is a strong argument for the model validity, because of links with the causality of filters. The random propagation time **A** is not sufficiently defined by its one-dimensional probability laws, and we need insights about two-dimensional laws. The difficulty lies in the fact that the order two moment of the r.v $A(t)$ does not exist, so that the problem is outside the familiar framework of stationary processes with finite autocorrelations and power spectra. In the appendix, we propose a solution to this drawback.

VIII. Appendix

1) Let assume that the r.v. (random variables) $X_n, n \in \mathbb{Z}$, are (mutually) independent obeing same probability stable laws with parameters $m = 0$ and any a, α, β defined by (15). Except when $\alpha = 2$ (the Gaussian), $\mathrm{E}\left[X_n^2\right] = \infty$. It is a drawback which forbids the use of mean-square tools. We define the r.v. Y from the sequence of positive real numbers $\mathbf{a} = \{a_k, k \in \mathbb{Z}\}$ by

$$Y = \sum_{k=-\infty}^{\infty} a_k X_k, a_k \geq 0.$$

From the independence of the X_k we have

$$\mathrm{E}\left[\exp\left(-\sum_{k=m}^{n} i\omega a_k X_k\right)\right] = \prod_{k=m}^{n} \mathrm{E}\left[\exp\left(-i\omega a_k X_k\right)\right] = \exp\left[-a|\omega|^\alpha c(\omega)\sum_{k=m}^{n} a_k^\alpha\right]$$

with $c(\omega) = 1 - ib\tan\frac{\pi\alpha}{2}\mathrm{sgn}\omega$ (for $\alpha \neq 1$). The Levy continuity theorem learns us that the r.v Y exists in the sense of the convergence in law if and only if [2]

$$\sum_{k=-\infty}^{\infty} a_k^\alpha < \infty.$$

In this situation the probability law of Y is stable with parameters

$$\left(0, a \sum_{k=-\infty}^{\infty} a_k^\alpha, \alpha, \beta\right).$$

When $|x| \to \infty$, the probability density of the X_k converges to 0 at least like $|x|^{-1-\alpha}$ [2]. This allows to assert the existence a.s (almost sure) of Y by application of the three-series theorem of Kolmogorov [1].

2) For any $h > 0$, we define $\mathbf{U}_h = \{U_h(t), t \in \mathbb{R}\}$ by

$$U_h(t) = h^{1/\alpha} \sum_{k=-\infty}^{\infty} f(t - kh) X_k \qquad (24)$$

where $f(t)$ is some positive, even and regular enough function, decreasing on $\mathbb{R}^+$. $U_h(t)$ is well defined provided that

$$\sum_{k=-\infty}^{\infty} f^\alpha(t - kh) < \infty.$$

The parameters of the stable law linked to $U_h(t)$ are

$$(0, \theta_h, \alpha, \beta)$$

where $\theta_h(t)$ is defined by

$$\theta_h(t) = ah \sum_{k=-\infty}^{\infty} f^\alpha(t - kh)$$

which can be chosen arbitrary close to the constant

$$\theta_0 = 2a \int_0^\infty f^\alpha(u)\, du \qquad (25)$$

when h is small enough and assuming the existence of the integral. In this case, $U_h(t)$ converges in law when $h \to 0$ to the stable law

$$(0, \theta_0, \alpha, \beta).$$

It is easy to construct simple examples which show that $U_h(t)$ does not converge a.s when $h \to 0$.

3) For $t = 2mh, \tau = 2nh > 0$, we can write (f is an even function)

$$U_h(t) - U_h(t - \tau) = h^{1/\alpha}[A - B]$$
$$A =$$
$$\sum_{l=0}^{\infty} [f((n-l)h) - f((n+l)h)] X_{l+2m-n}$$
$$B =$$
$$\sum_{l=0}^{\infty} [f((n-l)h) - f((n+l)h)] X_{-l+2m-n}$$

which leads to (the coefficients of the X_k in A and B are positive because $f(t)$ is assumed decreasing on $\mathbb{R}^+$)

$$\begin{cases} \mathrm{E}\left[e^{-i\omega[U_h(t)-U_h(t-\tau)]}\right] = \\ \exp\left[-2a\mu_h(\tau)|\omega|^\alpha\right] \\ \mu_h(\tau) = \\ \sum_{l=0}^{\infty}\left[f\left(\frac{\tau}{2}-lh\right) - f\left(\frac{\tau}{2}+lh\right)\right]^\alpha \end{cases}$$

$\mu_h(\tau)$ is independent of t. When $h \to 0$, we obtain

$$\begin{cases} \lim_{h\to 0} \mathrm{E}\left[e^{-i\omega[U_h(t)-U_h(t-\tau)]}\right] = \\ \exp\left[-2a\mu_0(\tau)|\omega|^\alpha\right] \\ \mu_0(\tau) = \\ \int_0^\infty \left[f\left(\frac{\tau}{2}-u\right) - f\left(\frac{\tau}{2}+u\right)\right]^\alpha du. \end{cases} \qquad (26)$$

Let note that the last probability law does not depend on t.

When $\alpha = 2$ (the Gaussian case), we obtain a wide class of characteristic functions but not the whole possibilities (Khinchine's criterion, see [2]). With this construction, it is possible to obtain a behavior nearby the origin point like

$$\begin{cases} \mu_0(\tau) = \left|\frac{\tau}{\tau_0}\right|^\gamma + o\left(|\tau|^\gamma\right) \\ \gamma = 1 \text{ when } 0 < \alpha < 1 \\ 1 < \gamma < \alpha \text{ when } 1 < \alpha < 2. \end{cases} \qquad (27)$$

REFERENCES

[1] M. Loeve, *Probability theory,* Van Nostrand, 1955.
[2] E. Lukacs, *Characteristic Functions,* Griffin, 1970.
[3] V. M. Zolotarev, *One-dimensional stable distributions,* Amer. Math. Soc. 1986.
[4] J. Nolan, *Stable Distributions,* Birkhauser Boston 2007.
[5] B. Lacaze, *A Stochastic Model for Acoustic Attenuation,* Waves in Random and Complex Media, **17** (3) (2007) 343-356.
[6] B. Lacaze, *A Stochastic Model for Propagation through Tissue,* IEEE Trans. Ultrason. Ferroelec. Freq. Contr. **56** (10) (2009) 2180-2186.
[7] B. Lacaze, *Random Propagation Times in Ultrasonics,* Waves in Random and Complex Media, **20** (1) (2010) 179-190.
[8] B. Lacaze, *Backscattering from Trees Explained by Random Propagation Times,* IEEE Trans. on Geoscience and Remote Sensing **50** (10) (2012) 4000-4005.
[9] B. Lacaze, *A Stochastic Model for Backscattering on the Sea,* Waves in Random and Complex Media, **16** (1) (2006) 69-74.
[10] B. Lacaze, *Weakenings and Redshifts Induced by Random Propagation Times,* Optics Com., **281** (10) (2008) 2740-2745.
[11] B. Lacaze, *Gaussian Delay Models for Light Broadenings and Redshifts,* Elec. J. of Theoretical Physics **6** (20) (2009) 385-398.
[12] B. Lacaze, *Equivalent random propagation time for coaxial cable*s, arXiv: 1209.4780 [physics.ins-det] 21 Sep 2012.
[13] B. Lacaze, *An unifying model for spectra of transmitted monochromatic waves,* Wave motion, **44** (2006) 70-76.
[14] B. Lacaze, *Stationary clock changes on stationary processes,* Signal Processing, **55** (2) (1996) 191-206.
[15] J. B. Billingsley, *Low-angle Radar Land Clutter,* William Andrew Pub. 2002.
[16] K. D. Ridley, *Measurements of laser phase fluctuations induced by atmospheric turbulence over 2 km and 17.5 km distances,* Applied Optics **50** (26) (2011) 5085-5092.
[17] B. Lacaze, *About the DC/AC Ratio of Radar Backscatter,* IEEE Trans. on Aerospace and Electronic Systems **50** (1) (2014) 364-370.
[18] R. B. Nelsen, *An Introduction to Copulas,* Springer Series in Statistics, 2006.
[19] L. E. Kinsler, A. R. Frey, A. B. Coppens, J. V. Sanders, *Fundamentals of Accoustics,* Wiley 4th ed. 2000.
[20] R. J. Urick, *Sound Propagation in the Sea,* DARPA Public Release, 1979.
[21] T. L. Szabo, *Causal theories and data for acoustic attenuation obeing a frequency power law,* J. Acoust. Soc. of Amer. **97** (1995) 14-24.
[22] P. He, *Experimental Verification of Models for Determining Dispersion from Attenuation,* IEEE Trans. on Ultrasonics, Ferroelectrics and Frequency Control, **46** (3) (1999) 706-714.
[23] W. P. Mason, *The propagation characteristics of sound tubes and acoustic filters,* Phys. Rev. **31** (1928) 283-295.
[24] J. Liu, X. Lin, *Equalization in High-Speed Communications Systems,* IEEE Circuits and Systems Magazine **4** (2) (2004) 4-17.
[25] N. Oussalah, Y. Zebboudj, S. A. Boggs, *Analytic Solutions for Pulse Propagation in Shielded Power Cable for Symmetric and Asymmetric PD Pulses,* IEEE Trans. Dielect. Elec. Insul **14** (5) (2007) 1264-1270.
[26] R. Papazyan, R. Eriksson, *Calibration for Time Domain Propagation Constant Measurements on Power Cables,* IEEE Trans. on Instrumentation and Measurement, **52** (2) (2003) 415-418.
[27] T. A. Andreeva, W. W. Durgin, *Ultrasound Technique for Prediction of Statistical Characteristics of Grid-Generated Turbulence,* AIAA Journal **41** (8) (2003) 1438-1443.
[28] D. Mintzer, *Wave propagation in a randomly inhomogeneous medium,* J. Acoust. Soc. of Amer. **25** (1953) 922-927.
[29] R. M. Narayanan, R. M. Doerr, D. C. Rundquist, *Temporal decorrelation of X-band backscatter from wind-influenced vegetation,* IEEE Trans. on Aerospace and Electronics Circuits, **28** (2) (1992) 404-412.

[30] G. Samorodnitsky, M. S. Taqqu, *Stable non-Gaussian random processes,* Chapman & Hall, 1994.

[31] B. Lacaze, *A random equivalent transit time for backscattering on trees,* Signal Processing **58** (2) (1997) 223-225.

[32] B. Lacaze, *Modeling the HF channel with Gaussian random delays,* Signal Processing **64** (2) (1998) 215-220.