About the Bidimensional Beer-Lambert Law

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Abstract

In acoustics, ultrasonics and in electromagnetic wave propagation, the crossed medium can be often modelled by a linear invariant filter (LIF) which acts on a wide-sense stationary process. Its complex gain follows the Beer-Lambert law i.e is in the form $\exp[-\alpha z]$ where z is the thickness of the medium and α depends on the frequency and on the medium properties. This paper addresses a generalization for electromagnetic waves when the beam polarization has to be taken into account. In this case, we have to study the evolution of both components of the electric field (assumed orthogonal to the trajectory). We assume that each component at z is a linear function of both components at 0. New results are obtained modelling each piece of medium by four LIF. They lead to a great choice of possibilities in the medium modelling. Particular cases can be deduced from works of R. C. Jones on deterministic monochromatic light.

keywords: linear filtering, polarization, Beer-Lambert law, random processes.

1 Introduction

1.1 The Beer-Lambert law

The Beer-Lambert law (B.L law) states that some positive quantity $A(0)$ at the input of some medium varies following the equation

$$A(z) = A(0)\, e^{-\alpha z} \tag{1}$$

where z is the covered distance and α is a parameter defined by the medium [6]. The equality (1) comes from the approximation

$$A(z+dz) - A(z) \approx -\alpha A(z)\, dz$$

which postulates that the evolution of $A(z)$ on a small thickness dz of the medium is proportional to $A(z)$ and to dz, with a coefficient $\alpha > 0$ which is defined by the medium. Then the differential equation $A'(z) = -\alpha A(z)$ which leads to (1). A more general method starts from the equality

$$A(0)\, A(z+z') = A(z)\, A(z') \tag{2}$$

whatever z, z'. Equivalently the quotient $A(z+z')/A(z)$ depends only on z', and any piece of the medium of length z has the same behavior. If we assume that $A(z)$ is a continuous function on $\mathbb{R}^+$, the only solution of (2) is (1) for some $\alpha \in \mathbb{C}$.

If we take $A(0) = e^{i\omega_0 t}$, (1) becomes ($\mathcal{R}[\alpha]$ and $\mathcal{I}[\alpha]$ stand for the real and imaginary parts of α)

$$A(z) = \exp\left[i\omega_0\left(t - \frac{z}{\omega_0}\mathcal{I}[\alpha]\right) - z\mathcal{R}[\alpha]\right]. \tag{3}$$

This means that the monochromatic wave $e^{i\omega_0 t}$ is delayed by $\frac{z}{\omega_0}\mathcal{I}[\alpha]$ and weakened by $\exp[-z\mathcal{R}[\alpha]]$ when crossing a thickness z in the medium.

Now we place ourselves from a signal processing perspective. We assume that a piece of medium of any thickness z is equivalent to a LIF (Linear Invariant Filter) $\mathcal{F}_z$ with complex gain $F_z(\omega)$ and that any piece of thickness $z+z'$ has the behavior of two filters in series $\mathcal{F}_z$ and $\mathcal{F}_{z'}$ [15], [9]. This means that whatever the frequency $\omega/2\pi$

$$F_{z+z'}(\omega) = F_z(\omega)\, F_{z'}(\omega). \tag{4}$$

Obviously we take $F_0 = 1$. What preceeds implies that for each ω it exists a complex $\alpha(\omega)$ such that

$$F_z(\omega) = e^{-z\alpha(\omega)}$$

By definition $F_z(\omega)\, e^{i\omega t}$ is the output of the filter $\mathcal{F}_z$ when the input is $e^{i\omega t}$. For such an input the power P_z at the output is

$$P_z = e^{-2z\mathcal{R}[\alpha(\omega)]} \tag{5}$$

(5) summarizes the Beer-Lambert law for wave propagation through a continuous medium, used in acoustics, ultrasonics and electromagnetics. $\alpha(\omega)$ gives the attenuation of the wave (by its real part) and the celerity of the wave (by its imaginary part). The Kramers-Kronig relation links the real and imaginary parts of $F_z(\omega)$ which constitute a pair of Hilbert transforms [17], [11]. (5) is matched to monochromatic waves. More generally when the stationary process $\mathbf{Z}_0 = \{Z_0(t), t \in \mathbb{R}\}$ is the input of $\mathcal{F}_z$, the output $\mathbf{Z}_z = \{Z_z(t), t \in \mathbb{R}\}$ verifies (E[..] stands for mathematical expectation or ensemble mean)

$$P_z = \mathrm{E}\left[|Z_z(t)|^2\right] = \int_{-\infty}^{\infty} e^{-2z\mathcal{R}[\alpha(\omega)]} s_0(\omega)\, d\omega$$

where $s_0(\omega)$ is the power spectral density of $\mathbf{Z}_0$ ($s_0(\omega) = \delta(\omega - \omega_0)$ for a unit power monochromatic wave at ω_0). The fact that measurements are generally performed by non-monochromatic waves lead to gaps with the Beer-Lambert law in the form (6) [2]. Also, B.L law can be untrue when powers are too high (which act on the medium properties) [1] or when the beam expands [10]. However, B.L law has many applications in physics, chemistry and medecine [5], [8], [4], [13].

1.2 A counter-example

The Beer-Lambert law applies for light amplitude or power after crossing continuous permanent media which can be viewed as a set of filters in series. However light or radar wave is not only defined by an amplitude [3], [14]. Electric field $\mathbf{E}^z = \left(\mathbf{E}_x^z, \mathbf{E}_y^z\right)$ is a two-dimensional vector with respect to axes Ox and Oy at distance z of origin O and with components

$$\mathbf{E}_x^z = \left\{E_x^z\left(t\right), t \in \mathbb{R}\right\}, \mathbf{E}_y^z = \left\{E_y^z\left(t\right), t \in \mathbb{R}\right\}.$$

It is assumed that the beam propagates in the neighbourood of the axis Oz and that the field is orthogonal to this axis. The beam is polarized in the direction θ at z when

$$\begin{cases} E_x^z\left(t\right) = A\left(t\right)\cos\theta \\ E_y^z\left(t\right) = A\left(t\right)\sin\theta \end{cases}$$

for some (real or complex) process $\mathbf{A} = \left\{A\left(t\right), t \in \mathbb{R}\right\}$. The beam is unpolarized at z when the cross-spectrum $s_{xy}^z\left(\omega\right)$ of components verifies

$$s_{xy}^z\left(\omega\right) = 0$$

whatever the basis Oxy. This implies the equality of power spectra. The wave is partially polarized in other cases.

The power P_z at z for a stationary wave is defined by

$$P_z = \mathrm{E}\left[\left|E_x^z\left(t\right)\right|^2\right] + \mathrm{E}\left[\left|E_y^z\left(t\right)\right|^2\right]. \tag{6}$$

We know that media act upon polarization and then can influence measurements, for instance in the case of antennas (generally matched to a particular polarization state). B.L law is not available for behavior of a given component of the electric field except particular cases. For instance, take the wave

$$E_x^0\left(t\right) = e^{i\omega_0 t}, \quad E_y^0\left(t\right) = 0$$

which propagates in a medium which rotates the beam by angle proportional to thickness z. We have (c is the celerity in the medium)

$$E_x^z\left(t\right) = e^{i\omega_0\left(t - z/c\right)}\cos\left[z\alpha\left(\omega_0\right)\right]$$

If we measure $E_x^z\left(t\right)$ the medium is equivalent to a filter of complex gain

$$F_z\left(\omega\right) = e^{-i\omega z/c\left(\omega\right)}\cos\left[z\alpha\left(\omega\right)\right]$$

when a particular direction is chosen ($F_z\left(\omega\right) = e^{-i\omega z/c}\sin\left[z\alpha\left(\omega\right)\right]$ for the orthogonal direction and c can depend on ω). The term $\alpha\left(\omega\right)$ takes into account a possible dependency of the rotation angle with the frequency. For an antenna which selects the component in a given direction at distance z, the B.L law is not verified.

1.3 The two-dimensional case

The B.L law is established for one-component monochromatic waves (i.e for waves defined by only one quantity depending on the time t and on the space coordinate z). They are time-functions in the form $ae^{i\omega_0 t}$ ($a \in \mathbb{C}$ depends on z). Quasi-monochromatic waves can be defined as random processes in the form

$$E^z\left(t\right) = A^z\left(t\right)e^{i\omega_0 t}$$

where $\mathbf{A}^z = \{A^z(t), t \in \mathbb{R}\}$ is stationary with a baseband spectrum which cancels outside $(-\omega_1, \omega_1)$ with $\omega_1/\omega_0 \ll 1$. The spectral band of $\mathbf{E}^z$ is included in the interval $(\omega_0 - \omega_1, \omega_0 + \omega_1)$.

In the two-components case (i.e for waves defined by two quantities $E_x^z(t), E_y^z(t)$ depending on the time t and on the space coordinate z) a quasi-monochromatic beam is defined by

$$E_x^z(t) = e^{i\omega_0 t} A_x^z(t), \quad E_y^z(t) = e^{i\omega_0 t} A_y^z(t) \tag{7}$$

where $\mathbf{A}_x^z, \mathbf{A}_y^z$ are stationary with stationary correlation and power spectrum inside $(-\omega_1, \omega_1)$. These properties remain whatever the chosen coordinates axes. It is reasonable to say that a wave is (purely) monochromatic when $\mathbf{A}_x^z, \mathbf{A}_y^z$ verify a relation in the form (for some constant k and real $B_z(t)$)

$$A_x^z(t) = k \cos B^z(t), \ A_y^z(t) = k \sin B^z(t)$$

$B_z(t)$ represents the orientation and $ke^{i\omega_0 t}$ the complex amplitude of the field. This means that its (complex) amplitude is purely monochromatic in the usual sense. When $\mathbf{B}^z$ is degenerate ($B^z(t)$ does not depend on t), the wave is polarized (at z). In other cases, the wave is partially polarized. When $\mathbf{E}_x^z$ and $\mathbf{E}_y^z$ have same power spectrum and are uncorrelated, the wave is unpolarized at z. Both properties are true in any orthogonal system. In optics, beams are often quasi-monochromatic and partially polarized but it is not always true, particulary in astronomy and communications. Polarized and unpolarized beams are convenient idealizations.

The concept of quasi-monochromatic wave is often bad-fitted even in the light domain. For instance Wolf-Rayet stars and B.L Lacertae have lines of relative width larger than few percents. Idem for LEDs (light-emitting diodes) with width larger than 10%. A more general model has to be taken in these cases.

We have seen that the B.L law for one-component beams can be proved using a decomposition of media by LIF in series. The aim of this paper is to generalize the B.L law in the two-components case, using signal theory and modelling media as more general circuits. Inputs and outputs of these circuits are stationary processes which represent the components of the field. In the following section we consider that each component of the field at a distance z is the sum of two LIF outputs. The field components at the origin point ($z = 0$) are the inputs of these LIF. This model allows to determine the shape of the LIF characteristics, i.e the B.L law for bi-dimensional beams.

Because two inputs and two outputs, 2x2 matrices of filters complex gains will be defined. In the years 1940, R. C. Jones had developed a "New Calculus for the Treatment of Optical Systems" [7]. It was based on 2x2 matrices which act in the time domain on purely monochromatic waves. Though formally results of Jones are very close to formulas of this paper, they do not address the same objects. Actually Jones papers do not mention Beer neither Lambert or Bouguer, the pioneer of this topic.

2 Two-dimensional Beer-Lambert law

We deal with a beam which propagates in direction Oz of the orthogonal trihedron Oxyz. The electric field $\mathbf{E}^z$ at time t is defined by its components

$E_x^z(t), E_y^z(t)$ on axes Ox and Oy at distance z of the origin O. We assume that the medium between u and $u+z$ is defined by a set of 2x2 “scattering matrice” $\mathbf{H}^z$ independent of u

$$\mathbf{H}^z = \begin{bmatrix} H_{11}^z & H_{12}^z \\ H_{21}^z & H_{22}^z \end{bmatrix}$$

where the $H_{jk}^z(\omega)$ depend on the frequency $\omega/2\pi$ and are complex gains of LIF $\mathcal{H}_{jk}^z$ such that

$$\begin{cases} E_x^{u+z}(t) = \mathcal{H}_{11}^z[\mathbf{E}_x^u](t) + \mathcal{H}_{12}^z[\mathbf{E}_y^u](t) \\ E_y^{u+z}(t) = \mathcal{H}_{21}^z[\mathbf{E}_x^u](t) + \mathcal{H}_{22}^z[\mathbf{E}_y^u](t) \end{cases} \tag{8}$$

Equivalently the electric field $\mathbf{E}^{z+u}$ at $z+u$ is linearly dependent on its value $\mathbf{E}^u$ at u [12]. The linearity is expressed by the set of the LIF $\mathcal{H}_{jk}^z$ which depend only on the medium. It is an obvious generalization of what was explained for the one-component waves propagation. Filters parameters depend only on the thickness of the considered medium. However results can be obtained without this hypothesis. Figure 1 shows equivalent circuits of (8).

Filters in series lead to multiplication of complex gains and filters in parallel to addition. The figure 2 summarizes the following equality

$$\mathbf{H}^{z+u} = \mathbf{H}^z\mathbf{H}^u. \tag{9}$$

(9) is equivalent to

$$\begin{cases} H_{11}^{z+u} = H_{11}^z H_{11}^u + H_{12}^z H_{21}^u \\ H_{12}^{z+u} = H_{11}^z H_{12}^u + H_{12}^z H_{22}^u \\ H_{21}^{z+u} = H_{21}^z H_{11}^u + H_{22}^z H_{21}^u \\ H_{22}^{z+u} = H_{21}^z H_{12}^u + H_{22}^z H_{22}^u \end{cases} \tag{10}$$

We assume that the derivatives $h_{jk}^0 = \frac{\partial}{\partial z} H_{jk}^0$ are finite. The first equation of (10) can be written as

$$\frac{H_{11}^{z+u} - H_{11}^z}{u} = H_{11}^z \frac{H_{11}^u - 1}{u} + H_{12}^z \frac{H_{21}^u}{u}$$

Obviously we have $H_{11}^0 = H_{22}^0 = 1$ and $H_{12}^0 = H_{21}^0 = 0$. When $u \to 0$, we obtain (similar operation is done in the other equations)

$$\begin{cases} h_{11}^z = H_{11}^z h_{11}^0 + H_{12}^z h_{21}^0 \\ h_{12}^z = H_{11}^z h_{12}^0 + H_{12}^z h_{22}^0 \\ h_{21}^z = H_{21}^z h_{11}^0 + H_{22}^z h_{21}^0 \\ h_{22}^z = H_{21}^z h_{12}^0 + H_{22}^z h_{22}^0. \end{cases} \tag{11}$$

The differential system can be split in two subsystems (equ.1+2 and equ.3+4). We assume that

$$\lim_{z\to\infty} H_{jk}^z = 0 \tag{12}$$

because any wave is evanescent in a passive medium. Two cases can be highlighted following the (complex) eigenvalues λ_1, λ_2 of the matrix

$$\begin{bmatrix} h_{11}^0 & h_{21}^0 \\ h_{12}^0 & h_{22}^0 \end{bmatrix}.$$

Solutions are in the form following that $\lambda_1 \neq \lambda_2$ or $\lambda_1 = \lambda_2 = \lambda$

$$H_{jk}^z = c_{jk1} e^{\lambda_1 z} + c_{jk2} e^{\lambda_2 z}$$
$$\text{or } H_{jk}^z = (c_{jk1} z + c_{jk2}) e^{\lambda z}.$$

The eigenvalues cannot cancel (or corresponding coefficients cancel). because (12) . Taking into account the initial conditions

$$H_{11}^0 = H_{22}^0 = 1 \text{ and } H_{12}^0 = H_{21}^0 = 0 \tag{13}$$

leads to two cases

Case 1: $\lambda_1 \neq \lambda_2$

By identification with (11) we obtain

$$\left\{ \begin{array}{l} H_{11}^z = \alpha e^{\lambda_1 z} + (1-\alpha) e^{\lambda_2 z} \\ H_{12}^z = \frac{-h_{12}^0}{\lambda_2 - \lambda_1} \left(e^{\lambda_1 z} - e^{\lambda_2 z} \right) \\ H_{21}^z = \frac{-h_{21}^0}{\lambda_2 - \lambda_1} \left(e^{\lambda_1 z} - e^{\lambda_2 z} \right) \\ H_{22}^z = (1-\alpha) e^{\lambda_1 z} + \alpha e^{\lambda_2 z} \\ \alpha = \frac{\lambda_2 - h_{11}^0}{\lambda_2 - \lambda_1} \end{array} \right. \tag{14}$$

where the λ_j, h_{jk}^0 can depend on frequency $\omega/2\pi$ but are independent of z and

$$\left\{ \begin{array}{r} \lambda_1 = \frac{1}{2} \left(h_{11}^0 + h_{22}^0 + \sqrt{\rho} e^{i\theta/2} \right) \\ \lambda_2 = \frac{1}{2} \left(h_{11}^0 + h_{22}^0 - \sqrt{\rho} e^{i\theta/2} \right) \\ \Delta = \left(h_{11}^0 - h_{22}^0 \right)^2 + 4 h_{12}^0 h_{21}^0 = \rho e^{i\theta}. \end{array} \right. \tag{15}$$

The eigenvalues have real parts strictly negative (to fulfill the condition (12)).

Case 2: $\lambda_1 = \lambda_2$

The solutions are given by the equalities

$$\left\{ \begin{array}{ll} H_{11}^z = (az+1) e^{\lambda z}, & H_{12}^z = h_{12}^0 e^{\lambda z} \\ H_{22}^z = (-az+1) e^{\lambda z}, & H_{21}^z = h_{21}^0 e^{\lambda z} \\ a = \frac{h_{11}^0 - h_{22}^0}{2}, & \lambda = \frac{h_{11}^0 + h_{22}^0}{2}. \end{array} \right. \tag{16}$$

The case $h_{11}^0 = h_{22}^0 \neq 0, h_{12}^0 = h_{21}^0 = 0$ leads to the usual B.L law. These equalities are verified in any system of coordinates. Components evolve independently, with same attenuation and celerity. This corresponds to a medium with all possible properties of symmetry.

In all cases, the real part of eigenvalues different from 0 have to be negative for passive media which weaken waves. Moreover, the solutions are only matched to equations $(11), (12), (13)$.

3 Examples

In examples, we assume that the parameters $h_{jk}^0(\omega)$ are constant on spectral supports of inputs $\mathbf{E}_x^0$,$\mathbf{E}_y^0$.

3.1 Example 1

Let assume that
$$h_{11}^0 \neq h_{22}^0 \text{ and } h_{12}^0 = h_{21}^0 = 0.$$
We are in the case 1 with
$$H_{11}^z = e^{h_{11}^0 z}, H_{22}^z = e^{h_{22}^0 z}, H_{12}^z = H_{21}^z = 0.$$
This means that a beam $e^{i\omega t}$ polarized on Ox is transmitted with weakening $\exp\left[\mathcal{R}\left[h_{11}^0\right] z\right]$ and delay $\mathcal{I}\left[h_{11}^0\right] z/\omega$. If polarized along Oy, the weakening is $\exp\left[\mathcal{R}\left[h_{22}^0\right] z\right]$ and the delay is $\mathcal{I}\left[h_{22}^0\right] z/\omega$. When $\mathcal{R}\left[h_{11}^0\right]/\mathcal{R}\left[h_{22}^0\right] \ll 1$, the first component disappears before the second component, independently of the values of $\mathcal{I}\left[h_{11}^0\right]$ and $\mathcal{I}\left[h_{22}^0\right]$ which define the refraction indices of the medium. Then we can give to h_{11}^0, h_{22}^0 values fitted to a dichroic material. More generally when
$$E_x^z(t) = e^{h_{11}^0 z} E_x^0(t), \; E_y^z(t) = e^{h_{22}^0 z} E_y^0(t)$$
both components evolve independently and the usual B.L law is verified for each component. The power P_z at z becomes (we have assumed that the h_{jk}^0 are constant with respect to frequency)
$$P_z = e^{2z\mathcal{R}\left[h_{11}^0\right]}\sigma_x^2 + e^{2z\mathcal{R}\left[h_{22}^0\right]}\sigma_y^2$$
where $\sigma_x^2 = \mathrm{E}\left[\left|E_x^0(t)\right|^2\right], \sigma_y^2 = \mathrm{E}\left[\left|E_y^0(t)\right|^2\right]$. We retrieve the usual result when $\mathcal{R}\left[h_{11}^0\right] = \mathcal{R}\left[h_{22}^0\right]$. In other cases usual B.L law is no longer valid because the two terms in P_z have different behaviors.

3.2 Example 2

Let assume that we are in the case 1 (two distinct eigenvalues λ_1, λ_2 different from 0) and that we deal with a monochromatic wave polarized along Ox:
$$E_x^0(t) = e^{i\omega_0 t}, \; E_y^0(t) = 0.$$
By (8) we have
$$\mathbf{E}_x^z = \mathcal{H}_{11}^z\left[\mathbf{E}_x^0\right], \; \mathbf{E}_y^z = \mathcal{H}_{21}^z\left[\mathbf{E}_x^0\right].$$
$\mathbf{E}^0$ is transformed in $\mathbf{E}^z$ defined by
$$\begin{cases} E_x^z(t) = \left[\frac{\lambda_2 - h_{11}^0}{\lambda_2 - \lambda_1} e^{\lambda_1 z} + \frac{h_{11}^0 - \lambda_1}{\lambda_2 - \lambda_1} e^{\lambda_2 z}\right] e^{i\omega_0 t} \\ E_y^z(t) = \frac{-h_{21}^0}{\lambda_2 - \lambda_1}\left(e^{\lambda_1 z} - e^{\lambda_2 z}\right) e^{i\omega_0 t} \end{cases}$$
where parameters can be complex. At z, both components are weakened and delayed through two terms functions of z ($e^{\lambda_1 z}$ and $e^{\lambda_2 z}$) and not one as in the usual B.L law.

1) When $h_{21}^0 = 0$ we have
$$E_x^z(t) = e^{h_{11}^0 z + i\omega_0 t}, \quad E_y^z(t) = 0.$$
$\mathbf{E}^z$ is polarized along Ox with $\mathcal{R}\left[h_{11}^0\right]$ and $\mathcal{I}\left[h_{11}^0\right]$ as parameters of weakening and of delay. $P_z = e^{2z\mathcal{R}\left[h_{11}^0\right]}$ has the shape (5) of the usual B.L law.

2) Except when $h^0_{21}=0$, the monochromatic wave $\mathbf{E}^z$ is no longer polarized. Assume that $h^0_{21}\neq 0, h^0_{12}=0$ (which implies $h^0_{11}\neq 0, h^0_{22}\neq 0, h^0_{11}\neq h^0_{22}$). From (14)

$$\left\{\begin{array}{l} E^z_x(t)=e^{h^0_{11}z+i\omega_0 t} \\ E^z_y(t)=\frac{h^0_{21}}{h^0_{11}-h^0_{22}}\left(e^{h^0_{11}z}-e^{h^0_{22}z}\right)e^{i\omega_0 t}. \end{array}\right.$$

Even when $\mathcal{R}\left[h^0_{11}\right]=\mathcal{R}\left[h^0_{22}\right]$ it is impossible to have P_z like (5). Consequently we understand using symmetries that usual BL law can be true for a particular polarization, and untrue for other polarizations.

3.3 Example 3

1) The case

$$h^0_{12}=-h^0_{21},\quad h^0_{11}=h^0_{22}\neq 0 \tag{17}$$

is particularly interesting. We have

$$\lambda_1=h^0_{11}+ih^0_{12},\quad \lambda_2=h^0_{11}-ih^0_{12}$$

and (14) becomes

$$\left\{\begin{array}{l} H^z_{11}=H^z_{22}=\frac{1}{2}\left(e^{\lambda_1 z}+e^{\lambda_2 z}\right) \\ H^z_{12}=-H^z_{21}=-\frac{i}{2}\left(e^{\lambda_1 z}-e^{\lambda_2 z}\right). \end{array}\right.$$

Now we assume that the electric field $\mathbf{E}^0$ is monochromatic and polarized at angle ϕ with respect to Ox

$$\left\{\begin{array}{l} E^0_x(t)=e^{i\omega_0 t}\cos\phi \\ E^0_y(t)=e^{i\omega_0 t}\sin\phi. \end{array}\right.$$

From (8) and after elementary algebra

$$\left\{\begin{array}{l} E^z_x(t)=e^{i\omega_0 t+h^0_{11}z}\cos\left(\phi-zh^0_{12}\right) \\ E^z_y(t)=e^{i\omega_0 t+h^0_{11}z}\sin\left(\phi-zh^0_{12}\right). \end{array}\right. \tag{18}$$

For real h^0_{12}, the electric field $\mathbf{E}^z$ at z is polarized at the angle $\left(\phi-zh^0_{12}\right)$. Then a monochromatic wave at the frequency $\omega_0/2\pi$ is rotated by the angle $-zh^0_{12}$ and attenuated by $\exp\left(z\mathcal{R}\left[h^0_{11}\right]\right)$. Moreover $-\frac{z}{\omega_0}\mathcal{I}\left[h^0_{11}\right]$ is the time spent by the wave between O and z which shows that the medium refraction index $n(\omega_0)$ is equal to (c is the light velocity in vacuum)

$$n(\omega_0)=-\frac{c}{\omega_0}\mathcal{I}\left[h^0_{11}\right].$$

The rotation angle is independent of the polarization angle ϕ. Because $P_z=e^{2z\mathcal{R}\left[h^0_{11}\right]}$ usual B.L law is obeyed for the amplitude and the power, but not for components.

2) When h^0_{12} is not real (18) is developed in

$$\left\{\begin{array}{l} E^z_x(t)=e^{i\omega_0 t+h^0_{11}z}(\cos a\cosh b+i\sin a\sinh b) \\ E^z_y(t)=e^{i\omega_0 t+h^0_{11}z}(\sin a\cosh b-i\cos a\sinh b) \\ \qquad a=\phi-z\mathcal{R}\left[h^0_{12}\right],\quad b=z\mathcal{I}\left[h^0_{12}\right]. \end{array}\right. \tag{19}$$

Consequently $\mathbf{E}^z$ has two components polarized in orthogonal directions of angles a and $\left(a+\frac{\pi}{2}\right)$ with respect to Ox and of "height" B_1^z and B_2^z such that

$$\left\{\begin{array}{c} B_1^z = e^{\mathcal{R}\left[h_{11}^0\right]z}\cosh\left(z\mathcal{I}\left[h_{12}^0\right]\right) \\ B_2^z = -ie^{\mathcal{R}\left[h_{11}^0\right]z}\sinh\left(z\mathcal{I}\left[h_{12}^0\right]\right) \\ \left|\frac{B_2^z}{B_1^z}\right| = \tanh\left|z\mathcal{I}\left[h_{12}^0\right]\right|. \end{array}\right. \tag{20}$$

$\left|\frac{B_2^z}{B_1^z}\right|$ increases from 0 to 1 when z increases from 0 to ∞. The main component is rotated by $-z\mathcal{R}\left[h_{12}^0\right]$ and attenuated by $\exp\left(z\mathcal{R}\left[h_{11}^0\right]\right)\cosh b$. In the same time a second component appears which is orthogonal to the first component. The ratio of "heights" increases with z and tends towards 1. The power P_z at z is given taking into account othogonality of components

$$P_z = e^{2z\mathcal{R}\left[h_{11}^0\right]}\cosh\left(2z\mathcal{I}\left[h_{12}^0\right]\right)$$

which shows that the usual B.L law is not valid (except in the case of real h_{12}^0 which leads to $P_z = e^{2z\mathcal{R}\left[h_{11}^0\right]}$). The power of each component $\mathbf{E}_x^z, \mathbf{E}_y^z$ does not follow the usual B.L law. For instance, for the power of the component $\mathbf{E}_x^z$

$$P_x^z = e^{2z\mathcal{R}\left[h_{11}^0\right]}\left(\cos^2\left(\phi - z\mathcal{R}\left[h_{12}^0\right]\right) + \sinh^2\left(z\mathcal{I}\left[h_{12}^0\right]\right)\right).$$

The same remark is true for components in the directions a and $\left(a+\frac{\pi}{2}\right)$ with respect to Ox.

3.4 Example 4

We assume that $\mathbf{E}^0$ is a quasi-monochromatic beam (see section 1)

$$E_x^0(t) = e^{i\omega_0 t}A_x^0(t), \quad E_y^0(t) = e^{i\omega_0 t}A_y^0(t).$$

The $h_{jk}^0(\omega)$ are constant with respect of ω on the beam spectrum. If the conditions (17) are fulfilled, the components $\mathbf{E}_x^0$,$\mathbf{E}_y^0$ are split by the medium in two orthogonal parts. With respect to $Ox'y'$ defined by

$$(Ox, Ox') = (Oy, Oy') = -z\mathcal{R}\left[h_{12}^0\right]$$

the beam at z verifies using (19) and (20)

$$\left\{\begin{array}{c} E_{x'}^z(t) = e^{\mathcal{R}\left[h_{11}^0\right]z}\left[A_x^0(t)\cosh\left(z\mathcal{I}\left[h_{12}^0\right]\right) + A_y^0(t)\sinh\left(z\mathcal{I}\left[h_{12}^0\right]\right)\right] \\ E_{y'}^z(t) = e^{\mathcal{R}\left[h_{11}^0\right]z}\left[A_y^0(t)\cosh\left(z\mathcal{I}\left[h_{12}^0\right]\right) - A_x^0(t)\sinh\left(z\mathcal{I}\left[h_{12}^0\right]\right)\right]. \end{array}\right.$$

We deduce the power P_z defined by (6)

$$\left\{\begin{array}{c} P_z = e^{2\mathcal{R}\left[h_{11}^0\right]z}\left[P_0\cosh\left(2z\mathcal{I}\left[h_{12}^0\right]\right) + \theta\sinh\left(2z\mathcal{I}\left[h_{12}^0\right]\right)\right] \\ \theta = 2\mathcal{I}\left\{\mathrm{E}\left[A_x^0(t)A_y^{0*}(t)\right]\right\}. \end{array}\right.$$

Whatever the polarization state of the beam at $z=0$, the usual B.L law is verified if and only if h_{12}^0 is real, i.e when the effect of the medium is a rotation (added to a weakening). Whatever h_{12}^0, we have $\theta = 0$ for instance when $\mathbf{E}^0$ is polarized (E[..] is real) or unpolarized ($\mathbf{E}_x^0$ and $\mathbf{E}_y^0$ are uncorrelated).

3.5 Example 5

We assume that $\mathbf{E}^0$ is a quasi-monochromatic beam like (7) and

$$h_{11}^0 = h_{22}^0, \quad h_{12}^0 = h_{21}^0$$

i.e transfers between the coordinates are symmetric. From (8) we have

$$\begin{cases} H_{11}^z = H_{22}^z = e^{h_{11}^0 z} \cosh h_{12}^0 z \\ H_{12}^z = H_{21}^z = e^{h_{11}^0 z} \sinh h_{12}^0 z. \end{cases}$$

However we remark that these relations are not maintained in other reference systems. For instance, in $Ox'y'$ with $(Ox, Ox') = \phi$ we have (the k_{ij}^0 are the new parameters)

$$\begin{cases} k_{11}^0 = h_{11}^0 + h_{12}^0 \sin 2\phi \\ k_{22}^0 = h_{11}^0 - h_{12}^0 \sin 2\phi \\ k_{12}^0 = k_{21}^0 = h_{12}^0 \cos 2\phi. \end{cases}$$

Elementary algebra leads to

$$\begin{cases} P_z = e^{2\mathcal{R}\left[h_{11}^0\right] z} \left[P_0 \cosh\left(2z\mathcal{R}\left[h_{12}^0\right]\right) + \theta' \sinh\left(2z\mathcal{R}\left[h_{12}^0\right]\right)\right] \\ \theta' = 2\mathcal{R}\left\{\mathrm{E}\left[A_x^0(t)\, A_y^{0*}(t)\right]\right\} \end{cases}$$

which proves that the usual B.L law is not true apart from particular cases.

4 Conclusion

The Beer-Lambert law (actually due to P. Bouguer around 1729) was firstly used to measure the concentration of solutions . It addresses the problem of concentration measurement of some kind of molecules in a liquid. In equation 1, we have

$$\alpha = k(\omega)\, a \tag{21}$$

where a is the concentration and $k(\omega)$ is a wavelength-dependent absorptivity coefficient. k is deduced from a measurement of the attenuation for a known value of a. The property of linearity with respect to the distance is due to Lambert and the linearity with respect to the concentration of absorbing species in the material was highlighted by Beer. The BL law intervenes in wave propagation to explain together the attenuation and the dispersion whatever the crossed medium.

A version of the Beer-Lambert law addresses the power as a function of the medium thickness, whatever the nature of the wave, acoustic or electromagnetic. The result has the form $A(z) = A(0)\, e^{-\alpha z}$ where α is a function of the medium and of the frequency $\omega/2\pi$. Very often α is a power function of ω. Its estimation has numerous applications in medecine [13], [16]. Also, chemistry uses the measurements of α because it is a linear function of the number of particles imbedded in the medium. The B.L law can be wrong for instance when multipaths in fibres or in case of too strong transmitted powers [1], [2]. Moreover the medium and the devices can be sensitive to polarization state. For instance the B.L law can be untrue in the case of birefringence for the medium and when devices select only one component of the field. However a generalization is possible studying separately both components of the electric field which defines an electromagnetic beam.

The B.L law is easily proved modelling a medium thickness as a linear invariant filter (LIF) where input and output show the evolution of the quantity of interest (for instance an amplitude or a power). In this paper we study the evolution of two quantities, the components of the electric field of an electromagnetic beam. To take into account interactions between components, a piece of medium is modelled by four LIF. We assume that the crossings of two successive medium pieces are independent events. This hypothesis suffices to determine the shape of the LIF complex gains. They are defined by four parameters h_{11}^0, h_{12}^0 , h_{21}^0 , h_{22}^0 which depend on the medium and which may depend on the frequency. This model generalizes the Jones matrices used in the deterministic monochromatic beam to stationary random beams. As soon explained, Jones papers do not contain comments about the BL law. Examples of section 3 show that the set of parameters can be fitted to realistic situations.

References

[1] H. Abitan, H. Bohr, P. Buchhave, *Correction to the Beer-Lambert-Bouguer law for optical absorption,* Applied Optics 47 (29) (10-2008) 5354-5357.

[2] H. C. Allen, T. Brauers, B. J. Finlayson-Pitts, *Illustrating Deviations in the Beer-Lambert Law in an Instrumental Analysis Laboratory: Measuring Atmospheric Pollutants by Differential Optical Absorption Spectrometry,* J. of Chemical Education 74 (12) (12-1997) 1459-1462.

[3] M. Born, E. Wolf, *Principles of Optics,* Pergamon Press 1975.

[4] B. T. Fischer, D. W. Hahn, *Measurement of Small-Signal Absorption Coefficient and Absorption Cross Section of Collagen at 193-nm Excimer Laser Light and the Role of Collagen in Tissue Ablation,* Applied Optics 43 (29) (2004) 5443-5451.

[5] J. Hodgkinson, D. Masiyano, R. P. Tatam, *Using integrating spheres as absorption cells: path length distribution and application of Beer's Law,* Applied Optics 48 (30) (2009) 5748-5758.

[6] A. Ishimaru, *Wave propagation and scattering in Random Media,* Prentice Hall, 1991.

[7] R. C. Jones, *A new calculus for the treatment of optical systems* VII. *Properties of the N-Matrices.* J. of Optical Soc. of Am. 38 (8) (1948) 671-685.

[8] C. Klinteberg, A. Pifferi, S. Andersson-Engels, R. Cubbedu, S. Svanberg, *In vivo spectroscopy of tumors sensitizers with femtosecond white light,* Applied Optics 44 (11) (2005) 2213-2220.

[9] B. Lacaze, *Stationary clock changes on stationary processes,* Signal Processing, 55 (2) (1996) 191-206.

[10] B. Lacaze, *Gaps of FSO beams with the Beer-Lambert law,* Applied Optics 48 (14) (2009) 2702-2706.

[11] B. Lacaze, *Random propagation times in ultrasonics,* Waves in Random and Complex Media 20 (1) (2010) 179-190.

[12] B. Lacaze, *Beams Propagation Modelled by Bi-filters,* El. J. of Theoretical Physics 7 (24) (2010) 171-196.

[13] P. A. Lewin, *Quo vadis medical ultrasound?* Ultrasonics 42 (4-2004) 1-5.

[14] L. Mandel, E. Wolf, *Optical coherence and quantum optics,* Cambridge University Press, 1995.

[15] A. Papoulis, *Probability, Random Variables, and Stochastic Processes,* McGraw-Hill, 1991.

[16] K. J. Parker, R. M. Lerner, R. C. Waag, *Attenuation of Ultrasound: Magnitude and Frequency Dependence for Tissue Characterization,* Radiology 153 (1984) 785-788.

[17] K. R. Waters, J. Mobley, J. G. Muller, *Causality-imposed (Kramers-Kronig) relationships between attenuation and dispersion,* IEEE Trans. Ultrason. Ferroelec. Freq. Contr. 52 (2005) 822-833.