Beams Propagation Modelled by Bi-filters

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Abstract

In acoustic, ultrasonic or electromagnetic propagation, crossed media are often modelled by linear filters with complex gains in accordance with the Beer-Lambert law. This paper addresses the problem of propagation in media where polarization has to be taken into account. Because waves are now bi-dimensional, an unique filter is not sufficient to represent the effects of the medium. We propose a model which uses four linear invariant filters, which allows to take into account exchanges between components of the field. We call it bi-filter because it has two inputs and two outputs. Such a circuit can be fitted to light devices like polarizers, rotators and compensators and to propagation in free space. We give a generalization of the Beer-Lambert law which can be reduced to the usual one in some cases and which justifies the proposed model for propagation of electromagnic beams in continuous media.

keywords: linear filtering, polarization, Beer-Lambert law, random processes.

1 Introduction

Propagation of acoustic or electromagnetic beams is often studied through linear differential equations with coefficients which depend on the medium characteristics. Because of the difficulty to estimate parameters and to give well-fitted equations, the medium is often taken to be a linear invariant filter (LIF) with given spectral gains. For instance, propagation of ultrasonics through biological tissues is taken to be a filter with complex gain $\exp\left[-\alpha\omega^{\beta}\right]$. In scanning, values of parameters allow diagnosis [3], [10], [7]. The same model is used for losses and dispersion of radiowaves in coaxial cables, free space or fiber optics [5], [9].

We consider beams reduced to a trajectory in some medium. Beams measurements are very often reduced to the values of quantities available at two points, one at the origin and one at the end point (for instance the amplitude or the power). Even if the beam is received in a antenna with a given area, the information is often added in a coaxial cable or in a wave guide, and reduced to a finite number of complex quantities. Moreover, comparisons about values at the transmitter and at the receiver give insights about the medium, and it is the aim of most devices used in practice (and this is true for waves in any frequency band, optics, radio, acoustics or ultrasonics). Actually, the main difference between sonics or ultrasonics waves and radio or optical waves is the polarization of the last one. Then they are defined by vectors instead of scalars. This paper highlights this difference.

We define a one-dimensional beam by an electric field $\overrightarrow{\mathbf{E}^z}=\left\{\overrightarrow{E^z}(t), t \in \mathbb{R}\right\}$ at each point of a trajectory. t stands for the time, z is the coordinate on the trajectory and $\overrightarrow{E^z}(t)$ is a vector orthogonal to the trajectory. In numerous situations, it is sufficient to consider the amplitude of $\overrightarrow{E^z}(t)$, which defines the power and the spectral content of the wave. Then, it is sufficient to model the medium as a linear filtering, as in the case of acoustic wave propagation. It is no longer the case when polarization phenomena have to be taken into account. We have to show the evolution of two components $E_x^z(t), E_y^z(t)$ of the electric field which defines the wave. Because both components can be linked, the behavior of one component at the time t can depend on the other component. In a linear model, this means that each component is the addition of linear filtering of itself and of the other component. Consequently, the medium has to be defined by a family of four filters. Nevertheless, we will talk about a bi-filter because the operation defines a linear way between two couples of processes, two inputs and two outputs. We place ourselves in a stationary frame where the processes are stationary (in the wide sense) with stationary correlations. The filters which model the medium will be time invariant with the acronym LIF for Linear (Time) Invariant Filters and will be defined by complex gains rather than impulse responses. They generalize the notion of "scattering matrix" used for monochromatic waves where each coordinate of the scattered wave is a linear combination of transmitted coordinates. Though power spectra of electromagnetic waves are the most often assumed in limited bands, the definitions and computations will be done for general spectra possibly with infinite support.

Section 1 of this paper addresses the definition and properties of bi-filters. Sections 2 and 3 apply results of section 1 to one-dimensional beams. Section 4 studies a generalization of the Beer-Lambert law which shows why a continuous medium can be modelled by a bi-filter. Appendix gives proofs of formulae.

2 Bi-filtering

1) We consider two (real or complex) stationary processes

$$\mathbf{X}_1 = \{X_1(t), t \in \mathbb{R}\}, \mathbf{Y}_1 = \{Y_1(t), t \in \mathbb{R}\}$$

with spectral density s_{X_1}, s_{Y_1} depending on the frequency $\omega/2\pi$, but we will omit the variable ω when the result is not ambiguous. Moreover we assume a stationary correlation between them which defines a cross-spectrum $s_{X_1Y_1}$ such that [1], [12]

$$\left\{\begin{array}{l} \mathrm{E}[X_1(t)X_1^*(t-\tau)] = \int_{-\infty}^{\infty} s_{X_1}(\omega)e^{i\omega\tau}d\omega \\ \mathrm{E}[Y_1(t)Y_1^*(t-\tau)] = \int_{-\infty}^{\infty} s_{Y_1}(\omega)e^{i\omega\tau}d\omega \\ \mathrm{E}[X_1(t)Y_1^*(t-\tau)] = \int_{-\infty}^{\infty} s_{X_1Y_1}(\omega)e^{i\omega\tau}d\omega \end{array}\right. \tag{1}$$

where E[..] stands for the mathematical expectation (or ensemble mean) and the superscript * stands for the complex conjugate.

Now, we define the processes $\mathbf{X}_2=\{X_2(t), t \in \mathbb{R}\}, \mathbf{Y}_2=\{Y_2(t), t \in \mathbb{R}\}$ by

$$\left\{\begin{array}{l} X_2(t) = X_1 * h_{11}(t) + Y_1 * h_{12}(t) \\ Y_2(t) = X_1 * h_{21}(t) + Y_1 * h_{22}(t) \end{array}\right. \tag{2}$$

where the h_{jk} can be considered as impulse responses of four linear invariant filters $\mathcal{H}_{jk}$ and $(.*.)$ stands for the convolution product. For example we have

$$X_1 * h_{21}(t) = \int_{-\infty}^{\infty} X_1(u)h_{21}(t-u)\,du$$

We will say that $(\mathbf{X}_1, \mathbf{Y}_1)$ and $(\mathbf{X}_2, \mathbf{Y}_2)$ are the input and the output of the bi-filter $\mathcal{H} = \{\mathcal{H}_{jk}, j, k = 1, 2\}$. We know that it is more convenient to use spectral gains $H_{jk}(\omega)$ rather than impulse responses. When impulse responses are sufficiently regular we have together

$$\left\{\begin{array}{l} h_{jk}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H_{jk}(\omega)e^{i\omega t}d\omega \\ H_{jk}(\omega) = \int_{-\infty}^{\infty} h_{jk}(t)e^{-i\omega t}dt \end{array}\right. \tag{3}$$

but the h_{jk} are not always ordinary functions. The following writing of (2) is more general because the impulse responses do not appear

$$\left\{\begin{array}{l} X_2(t) = \mathcal{H}_{11}[\mathbf{X}_1](t) + \mathcal{H}_{12}[\mathbf{Y}_1](t) \\ Y_2(t) = \mathcal{H}_{21}[\mathbf{X}_1](t) + \mathcal{H}_{22}[\mathbf{Y}_1](t) \end{array}\right. \tag{4}$$

It is proved in appendix 1 that the bi-dimensional process $(\mathbf{X}_2, \mathbf{Y}_2)$ is stationary with spectral characteristics perfectly defined by formulae (34) which

Figure 1:

are simplified in

$$\begin{cases} s_{X_2Y_2} = H_{12}H_{22}^{*}s_{Y_1} + H_{11}H_{21}^{*}s_{X_1} + H_{12}H_{21}^{*}s_{Y_1X_1} + H_{11}H_{22}^{*}s_{Y_1X_1}^{*} \\ s_{X_2} = \left|H_{12}\right|^2 s_{Y_1} + \left|H_{11}\right|^2 s_{X_1} + 2\mathcal{R}\left[H_{12}H_{11}^{*}s_{Y_1X_1}\right] \\ s_{Y_2} = \left|H_{21}\right|^2 s_{X_1} + \left|H_{22}\right|^2 s_{Y_1} + 2\mathcal{R}\left[H_{22}H_{21}^{*}s_{Y_1X_1}\right] \\ s_{X_1X_2} = H_{12}^{*}s_{Y_1X_1}^{*} + H_{11}^{*}s_{X_1} \\ s_{X_1Y_2} = H_{22}^{*}s_{Y_1X_1}^{*} + H_{21}^{*}s_{X_1} \\ s_{Y_1X_2} = H_{11}^{*}s_{Y_1X_1} + H_{12}^{*}s_{Y_1} \\ s_{Y_1Y_2} = H_{21}^{*}s_{Y_1X_1} + H_{22}^{*}s_{Y_1} \end{cases} \tag{5}$$

where $\mathcal{R}\left[..\right]$ stands for the real part, s_a and s_{ab} are for spectra or cross-spectra, H_{jk} for filters complex gains, and all terms depend on ω. The figure 1 gives the scheme of the bi-filter. When used in the matrix form $\mathbf{H}$=$\left[H_{ij}\right]$, the bi-filter is a "scattering matrix" for each value of ω.

3 Application to waves

3.1 General formulae

1) We consider a beam which crosses a medium along the axis Oz of the orthogonal trihedron Oxyz. The beam is defined by its electrical field $\overrightarrow{E^z}(t) = \left(E_x^z(t), E_y^z(t)\right)$ at time t and distance z, where components (which are orthogonal to Oz) are taken on the axes Ox and Oy. We assume

a) that the processes (at O) $\mathbf{E}_x^0 = \left\{E_x^0(t), t \in \mathbb{R}\right\}$ and $\mathbf{E}_y^0 = \left\{E_y^0(t), t \in \mathbb{R}\right\}$ are stationary and with stationary correlation. The spectra and the cross-spectrum are

$$s_x^0(\omega), s_y^0(\omega), s_{xy}^0(\omega)$$

b) and that the medium between the points z_1 and z_2 can be characterized by four LIF (linear invariant filters) $\mathcal{H}_{jk}^{z_1 z_2}$ with the complex gains $H_{jk}^{z_1 z_2}(\omega)$ such that, whatever $0 \le u \le z$

$$\left\{\begin{array}{l} E_x^z(t) = \mathcal{H}_{11}^{uz}\left[\mathbf{E}_x^u\right](t) + \mathcal{H}_{12}^{uz}\left[\mathbf{E}_y^u\right](t) \\ E_y^z(t) = \mathcal{H}_{21}^{uz}\left[\mathbf{E}_x^u\right](t) + \mathcal{H}_{22}^{uz}\left[\mathbf{E}_y^u\right](t). \end{array}\right. \tag{6}$$

This writing highlights the dependence of $E_x^z(t)$ and $E_y^z(t)$ on the whole processes $\mathbf{E}_x^u$ and $\mathbf{E}_y^u$ and not only on the r.v. $E_x^u(t)$ and $E_y^u(t)$. Formulae (5) allow to write the different spectra and cross-spectra of the beam as (with $H_{jk} = H_{jk}^{0z}$ to alleviate formulae)

$$\left\{\begin{array}{l} s_{xy}^z = H_{12}H_{22}^* s_y^0 + H_{11}H_{21}^* s_x^0 + H_{12}H_{21}^* s_{yx}^0 + H_{11}H_{22}^* s_{xy}^0 \\ s_x^z = \left|H_{12}\right|^2 s_y^0 + \left|H_{11}\right|^2 s_x^0 + 2\mathcal{R}\left[H_{12}H_{11}^* s_{yx}^0\right] \\ s_y^z = \left|H_{21}\right|^2 s_x^0 + \left|H_{22}\right|^2 s_y^0 + 2\mathcal{R}\left[H_{22}H_{21}^* s_{yx}^0\right] \\ s_{xx}^{0z} = H_{12}^* s_{xy}^0 + H_{11}^* s_x^0 \\ s_{xy}^{0z} = H_{22}^* s_{xy}^0 + H_{21}^* s_x^0 \\ s_{yx}^{0z} = H_{11}^* s_{yx}^0 + H_{12}^* s_y^0 \\ s_{yy}^{0z} = H_{21}^* s_{yx}^0 + H_{22}^* s_y^0 \end{array}\right. \tag{7}$$

where the seven equalities are respectively for the couples (from the top to the bottom)

$$\left(E_x^z, E_y^z\right), \left(E_x^z, E_x^z\right), \left(E_y^z, E_y^z\right), \left(E_x^0, E_x^z\right), \left(E_x^0, E_y^z\right), \left(E_y^0, E_x^z\right), \left(E_y^0, E_y^z\right).$$

The power P_z of the wave at z is defined from the components by

$$P_z = \mathrm{E}\left[\left|E_x^z(t)\right|^2 + \left|E_y^z(t)\right|^2\right] = \int_{-\infty}^{\infty}\left[s_x^z + s_y^z\right](\omega)\, d\omega. \tag{8}$$

This definition is justified (at a multiplicative constant in accordance with some physical system of units) because devices for intensity measurements are not sensitive to the direction of the electric field. Also, the definition is independent of the basis used (see below).

2) In the proposed model, the filters complex gains are proper to the coordinate system Oxy. For the system Ox'y' deduced by a rotation of angle θ, the new system of complex gains $K_{jk}^{z_1 z_2}$ is defined by (we give the system for $K_{jk}^{0z} = K_{jk}$)

$$\begin{bmatrix} K_{11} \\ K_{12} \\ K_{21} \\ K_{22} \end{bmatrix} = \mathbf{\Theta} \begin{bmatrix} H_{11} \\ H_{12} \\ H_{21} \\ H_{22} \end{bmatrix}$$

$$\Theta = \begin{bmatrix} \mathbf{P}\cos\theta & \mathbf{P}\sin\theta \\ -\mathbf{P}\sin\theta & \mathbf{P}\cos\theta \end{bmatrix}, \mathbf{P} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}. \tag{9}$$

$\mathbf{\Theta}$ is orthogonal and then $\mathbf{\Theta}^{-1} = \mathbf{\Theta}^{t}$ (the transpose of $\mathbf{\Theta}$). Also, the only set of parameters which are invariant by rotation verify (see appendix 2)

$$H_{11} = H_{22}, \quad H_{12} = -H_{21}. \tag{10}$$

This set of H_{jk} defines the proper subspace of the eigenvalue 1. This property is very important and will often be used. Moreover it is easy to verify that

$$\begin{cases} H_{11} + H_{22} = K_{11} + K_{22} \\ H_{11}H_{22} - H_{12}H_{21} = K_{11}K_{22} - K_{12}K_{21}. \end{cases} \tag{11}$$

This property of invariance will be used in section 4.2.

3) The sum of bi-filters $\mathcal{K} = \mathcal{H} + \mathcal{H}'$ is naturally defined by $K_{jk} = H_{jk} + H'_{jk}$ whatever j, k. It is the same as the usual sum for linear filters (filters in parallel of the circuit theory). The product of bi-filters $\mathcal{K} = \mathcal{H}\text{x}\mathcal{H}'$ is defined in the same way as the filters in series of the circuit theory

$$(\mathbf{X}_2, \mathbf{Y}_2) = \mathcal{K}\left[\mathbf{X}_1, \mathbf{Y}_1\right] = \mathcal{H}'\left\{\mathcal{H}\left[\mathbf{X}_1, \mathbf{Y}_1\right]\right\}$$

The scheme of this operation is given in figure 2 with the notations used in the next sections. Equivalently, we have (because the complex gains of LIF in parallel and LIF in series are the sum and the product of individual complex gains)

$$\begin{cases} K_{11} = H_{11}H'_{11} + H_{21}H'_{12} \\ K_{12} = H_{12}H'_{11} + H_{22}H'_{12} \\ K_{21} = H_{11}H'_{21} + H_{21}H'_{22} \\ K_{22} = H_{12}H'_{21} + H_{22}H'_{22}. \end{cases}$$

This operation is associative but not commutative (which is a huge difference with ordinary LIF). The filter $\mathcal{I}$ such that

$$I_{11} = I_{22} = 1, I_{12} = I_{21} = 0$$

is the unit: $\mathcal{I}\text{x}\mathcal{H} = \mathcal{H}\text{x}\mathcal{I} = \mathcal{H}$.

Sums of bi-filters have to be used when the beam is split so that the results cross different media. Each of them corresponds to a bi-filter and outputs are added (similar to a fringes pattern). The product is for a beam which crosses two successive media, or two successive thickness of the same medium, each of them being defined as a bi-filter.

3.2 Polarized beam

A "polarized wave" at z is defined by a direction ψ (with respect to the axis Ox), and some stationary process $\mathbf{A}^z$=$\{A^z(t), t \in \mathbb{R}\}$

$$\begin{cases} E_x^z(t) = A^z(t)\cos\psi \\ E_y^z(t) = A^z(t)\sin\psi. \end{cases} \tag{12}$$

This definition is consistent because, in the basis Ox'y' such as (Ox, Ox') = ψ' we have

$$\left\{ \begin{array}{l} E^z_{x'}(t) = A^z(t)\cos(\psi - \psi') \\ E^z_{y'}(t) = A^z(t)\sin(\psi - \psi'). \end{array} \right.$$

In this paper the "polarized wave" is for the "linear polarized wave" used for deterministic beams. Because the sum of two polarized waves $(\mathbf{A}^z, \psi)$ and $(\mathbf{B}^z, \phi)$ is generally not polarized, the set of polarized waves has not an interesting algebraic structure. But, such waves can be treated separately when transformations are linear. It is worth-noting that the wave

$$\mathcal{F}\left[\mathbf{E}^z_x\right], \mathcal{F}\left[\mathbf{E}^z_y\right]$$

where $\mathcal{F}$ is any LIF, is polarized in the same direction that $\left(\mathbf{E}^z_x, \mathbf{E}^z_y\right)$.

From (11) , a necessary condition for a polarized wave is

$$\rho^z_x \rho^z_y = \left(\rho^z_{xy}\right)^2 \tag{13}$$

where

$$\rho^z_x = \mathrm{E}\left[\left|E^z_x(t)\right|^2\right], \rho^z_y = \mathrm{E}\left[\left|E^z_y(t)\right|^2\right], \rho^z_{xy} = \mathrm{E}\left[E^z_x(t)\, E^{z*}_y(t)\right]$$

(and not $\left|\rho^z_{xy}\right|^2$ except for real processes). Conversely, if (13) is verified, ρ^z_{xy} is real (because $\rho^z_x \rho^z_y \geq 0$), which leads to

$$E\left[\left|E^z_x(t) - \lambda E^z_y(t)\right|^2\right] = \left|\sqrt{\rho^z_x} - \lambda\varepsilon\sqrt{\rho^z_y}\right|^2$$

where $\varepsilon = \pm 1$ and $\lambda \in \mathbb{C}$ ($\varepsilon = 1$ when $\rho_{xy} \geq 0$). Consequently, we have $E^z_x(t) = \lambda E^z_y(t)$ with $\lambda = \varepsilon\sqrt{\rho^z_x / \rho^z_y}$ (quantity which is real and independent of t) and (13) becomes a sufficient condition for polarization. From (12) the condition (13) is true whatever the system Oxy. Reciprocally, when verified in one particular system, it is verified in the others. Also, the equality (13) is true for correlations and then for (regular) spectra

$$s^z_x s^z_y = \left(s^z_{xy}\right)^2 \tag{14}$$

but this last equality is not sufficient for polarization. As an example

$$\left\{ \begin{array}{l} E^z_x(t) = A^z(t)\cos\psi \\ E^z_y(t) = B^z(t)\sin\psi \end{array} \right.$$

where $\mathbf{B}^z$ is the output of a LIF with input $\mathbf{A}^z$ and complex gain $\varepsilon(\omega)$ taking only the values ± 1. The condition (14) is fulfilled but the wave is not generally polarized and can be split in the sum of two polarized waves in the directions ψ and $-\psi$.

3.3 Unpolarized beam

1) We define an "unpolarized wave" $\overrightarrow{\mathbf{E}^z}$ at z by the condition (whatever the systems Oxy, Ox'y')

$$s^z_{xy} = s^z_{x'y'} = 0.$$

Equivalently (see appendix 3)

$$\left\{ \begin{array}{l} s^z_x = s^z_y = s^z_{x'} = s^z_{y'} \\ s^z_{xy} = s^z_{x'y'} = 0. \end{array} \right. \tag{15}$$

The term "unpolarized wave" is used in this paper rather than the term "circular polarized wave" which we encounter for deterministic waves. The subset of uncorrelated unpolarized waves is a group for the addition. But the sum of two correlated unpolarized waves can be not unpolarized. For example, $(\mathbf{E}_x, \mathbf{E}_y)$ added to $(\mathbf{E}_x, -\mathbf{E}_y)$ is polarized along Ox.

In [15], pp. 350, and in the optics communauty, the unpolarized light (natural light) is defined by the equality $\rho^z_{xy} = 0$, whatever Oxy, which implies $\rho^z_x = \rho^z_y$. This condition is much more weak than the condition (15) . In the strong definition, an unpolarized wave remains unpolarized after crossing of a compensator. It is not true when using the weak definition.

2) $\overrightarrow{\mathbf{E}^0}$ is an unpolarized wave when its components $\mathbf{E}^0_x$ and $\mathbf{E}^0_y$ are uncorrelated whatever the system Oxy (strong definition). It is the case if and only if the components are uncorrelated for a given system in which the spectra are identical ($s^0_{xy} = 0, s^0_x = s^0_y$). In this circumstance, (7) is reduced to

$$\left\{ \begin{array}{l} s^z_{xy} = \left(H_{12}H^*_{22} + H_{11}H^*_{21}\right) s^0_x \\ s^z_x = \left(|H_{12}|^2 + |H_{11}|^2\right) s^0_x \\ s^z_y = \left(|H_{21}|^2 + |H_{22}|^2\right) s^0_x \end{array} \right.$$

Consequently the wave $\overrightarrow{\mathbf{E}^z}$ remains unpolarized whatever s^0_x if and only if

$$\left\{ \begin{array}{l} H_{12}H^*_{22} + H_{11}H^*_{21} = 0 \\ |H_{12}|^2 + |H_{11}|^2 = |H_{21}|^2 + |H_{22}|^2 . \end{array} \right.$$

Equivalently it exists a real function $P(\omega)$ such that

$$\left\{ \begin{array}{l} H_{22}(\omega) = H^*_{11}(\omega)\, e^{iP(\omega)} \\ H_{12}(\omega) = -H^*_{21}(\omega)\, e^{iP(\omega)}. \end{array} \right. \tag{16}$$

3.4 Partially polarized beam

Any wave which is not polarized or not unpolarized is a partially polarized wave. This is a definition which results in different classes following the definition given to unpolarization (weak or strong sense). In the class of beams defined in appendix 5, a partially polarized beam corresponds to a probability law different of the uniform on $(0, 2\pi)$ (case of an unpolarized wave) and not degenerate (polarized wave when degenerate).

The Stokes decomposition theorem is still a studied problem [19]. It states that a partially polarized beam is the sum of a polarized beam and of an unpolarized beam. For stationary processes with any spectra, solutions of the problem can be found according to the set where solutions are searched, and according to definition of unpolarization [8]. It is possible to construct waves which are polarized in more than one direction. Provided that the operations are linear, we can split the beam in a convenient number of polarized ones and we can study them separately.

3.5 Examples of bi-filters

We characterize the following elementary bi-filters by the matrix $\mathbf{H}$ of the $H_{jk}(\omega)$ in Oxy or the matrix $\mathbf{K}$ of the $K_{jk}(\omega)$ in Ox'y':

$$\mathbf{H} = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}$$

The elementary operations which follow are often assumed independent of the frequency. Actually this property is generally approached in the (limited) frequency band of an experiment. However the matrix $\mathbf{H}$ which defines the bi-filter is a function of the frequency $\omega/2\pi$ in most cases.

1) A compensator (θ_x, θ_y) between O and z induces different delays θ_x and θ_y for the components: $E_x^z(t) = E_x^0(t - \theta_x)$ and $E_y^z = E_y^0(t - \theta_y)$ [18]. The equivalent bi-filter is defined by

$$\mathbf{H} = \begin{bmatrix} e^{-i\omega\theta_x} & 0 \\ 0 & e^{-i\omega\theta_y} \end{bmatrix}$$

In the basis Ox'y' defined by ψ =(Ox, Ox'), we have

$$\mathbf{K} = \begin{bmatrix} e^{-i\omega\theta_x}\cos\psi & e^{-i\omega\theta_y}\sin\psi \\ -e^{-i\omega\theta_x}\sin\psi & e^{-i\omega\theta_y}\cos\psi \end{bmatrix}$$

2) The polarizer along the axis Ox' suppresses the orthogonal component along Oy'. This defines the bi-filter (in the basis Oxy with ψ = (Ox,Ox'))

$$\mathbf{H} = \begin{bmatrix} \cos^2\psi & \sin\psi\cos\psi \\ \sin\psi\cos\psi & \sin^2\psi \end{bmatrix}$$

3) The compensator (θ_x, θ_y) followed by a polarizer in the direction Ox' defines the bi-filter

$$\mathbf{H} = \begin{bmatrix} e^{-i\omega\theta_x}\cos^2\psi & e^{-i\omega\theta_y}\sin\psi\cos\psi \\ e^{-i\omega\theta_x}\sin\psi\cos\psi & e^{-i\omega\theta_y}\sin^2\psi \end{bmatrix}$$

The power P^z at z is given by, from (7) and (8)

$$\left\{\begin{array}{l} P^z = \int_{-\infty}^{\infty} \left[\alpha s_x^0 + \beta s_y^0 + 2\mathcal{R}\left(\gamma s_{yx}^0\right)\right](\omega)\, d\omega \\ \alpha = \cos^2\psi, \quad \beta = \sin^2\psi \\ \gamma = e^{-i\omega(\theta_y - \theta_x)} \sin\psi \cos\psi. \end{array}\right.$$

P^z is a function of $\theta_y - \theta_x$ and ψ. A good choice of these parameters allows to measure particular values of P^z. If the parameters do not depend on the frequency, this allows to estimate the quantities

$$\left\{\begin{array}{l} \mathrm{E}\left[\left|E_x^0(t)\right|^2\right] = \int_{-\infty}^{\infty} s_x^0(\omega)\, d\omega \\ \mathrm{E}\left[\left|E_y^0(t)\right|^2\right] = \int_{-\infty}^{\infty} s_y^0(\omega)\, d\omega \\ \mathrm{E}\left[E_x^0(t)\, E_y^{0*}(t)\right] = \int_{-\infty}^{\infty} s_{xy}^0(\omega)\, d\omega \end{array}\right.$$

which define the Stokes parameters and others [15].

3.6 Rotator

In appendix 2, we prove that any polarized beam in the direction ψ (with respect to Oxy) at O will be polarized in the direction $\psi + \theta$ at z ($\theta \neq \frac{\pi}{2} \bmod \pi$) if and only if

$$\left\{\begin{array}{l} H_{21}\cos\theta = H_{11}\sin\theta \\ H_{12} = -H_{21} \\ H_{11} = H_{22} \end{array}\right. \tag{17}$$

The angle of polarization is changed by the quantity θ. Moreover, the amplitudes at O and z are linked through a LIF with complex gain $H_{11}/\cos\theta$

$$\mathbf{A}^z = \frac{1}{\cos\theta}\mathcal{H}_{11}\left[\mathbf{A}^0\right]. \tag{18}$$

Note that the relations (17) and (18) are true whatever the axes, because using (8) and (9), they imply $K_{jk} = H_{jk}$. If we want that the amplitude remains unchanged, it is necessary that

$$H_{11} = \cos\theta$$

Here, the "amplitude" is the coordinate along the axis of polarization, and it is a quantity which can be complex.

More generally, any beam is defined by two polarized waves, the first one in the direction Ox, the second one in the direction Oy. Each of them is submitted to a rotation of angle θ. The amplitude of each of them is the result of a LIF with complex gain $H_{11}/\cos\theta$ (which is a quantity invariant in a change of basis). Then we can speak about a rotation whatever the kind of wave, polarized or not.

Also we see that an unpolarized wave will be changed by the bi-filter in an unpolarized wave such that (when $\theta \neq \frac{\pi}{2} \bmod \pi$)

$$s_x^z = s_y^z = |H_{11}|^2 \frac{s_x^0}{\cos^2\theta}.$$

But more general conditions allow to retain the unpolarization.

3.7 Depolarization

The medium in free space or in optical fibers is a source of depolarization. Bi-filters take into account this situation. If we start from a polarized wave along Ox, we have by definition

$$\begin{cases} E_x^z(t) = \mathcal{H}_{11}\left[\mathbf{E}_x^0\right](t) \\ E_y^z(t) = \mathcal{H}_{21}\left[\mathbf{E}_x^0\right](t) \end{cases}$$

Except when $[H_{11}/H_{21}](\omega)$ is a real constant, the wave is no longer polarized (assuming H_{11} and H_{21} different of 0). With $\rho_x^z, \rho_y^z, \rho_{xy}^z$ as defined in section 3.2, we define the constants a, b, c, d by

$$\begin{cases} c = 1 - a = e^{i\nu}\rho_x^z/\rho' \\ d = -b = -e^{i\nu}\rho_{xy}^z/\rho' \\ \rho' = \sqrt{\rho_x^z\rho_y^z - \left|\rho_{xy}^z\right|^2} \end{cases}$$

assuming that $\rho' \neq 0$ and where ν is any real number. This leads to the Stokes decomposition [19]

$$\begin{cases} E_x^z(t) = A(t) + B(t) \\ A(t) = aE_x^z(t) + bE_y^z(t) \\ B(t) = cE_x^z(t) + dE_y^z(t) \end{cases}$$

with a polarized part $(\mathbf{A}, \mathbf{0})$ and an unpolarized part (in the weak sense) $\left(\mathbf{B}, \mathbf{E}_y^z\right)$, which verify the conditions

$$\mathrm{E}\left[B(t)\,E_y^{z*}(t)\right] = 0, \quad \mathrm{E}\left[|B(t)|^2\right] = \mathrm{E}\left[\left|E_y^z(t)\right|^2\right].$$

To perform this decomposition with a given power for each part, it suffices to choose filters with complex gains H_{11}, H_{21} such that

$$\begin{cases} \rho_y^z = \int_{-\infty}^{\infty}\left[|H_{21}|^2 s_x^0\right](\omega)\,d\omega \\ \rho_x^z = \int_{-\infty}^{\infty}\left[|H_{11}|^2 s_x^0\right](\omega)\,d\omega \\ \rho_{xy}^z = \int_{-\infty}^{\infty}\left[H_{11}H_{21}^* s_x^0\right](\omega)\,d\omega. \end{cases}$$

The condition $\rho' \neq 0$ can always be verified using well-chosen filters. But other Stokes decompositions can be done, changing the definition or the basis [8].

3.8 A particular class of beams

The main problem is to know if the model is well-fitted to physical situations. A particular model is treated in appendix 5. A quasi-monochromatic light can be polarized, or unpolarized or partially polarized. In the first case, the electric field has a constant direction and in the second case, the electric field takes any direction with equal probability (independently with its amplitude). They are extreme situations and the intermediary situation is for a partially polarized light where the field direction is a random process with a one-dimensional probability law which is not degenerate (such as polarized light) neither uniformly distributed on $(0, 2\pi)$ (such as unpolarized light). A large class of light spectra verifies (see appendix 5)

$$\left\{ \begin{array}{l} s_x^0 (\omega) = \alpha \left| f (\omega) \right|^2 \\ s_y^0 (\omega) = (1 - \alpha) \left| f (\omega) \right|^2 \\ s_{xy}^0 (\omega) = \beta \left| f (\omega) \right|^2 \end{array} \right. \tag{19}$$

where $0 \leq \alpha \leq 1$, and $|\beta| \leq \sqrt{\alpha (1 - \alpha)}$. The values $|\beta| = \sqrt{\alpha (1 - \alpha)}$ are for polarized light in a direction defined by α, and $\alpha = \frac{1}{2}, \beta = 0$ for unpolarized light. A quasi-monochromatic wave corresponds to $f (\omega) = 0$ outside a short interval centered at some frequency $\omega_0 / 2\pi$. (7) shows that the bi-filter does not change the membership to the class if the parameters H_{jk} are constant on the spectral support of the wave. For instance, the wave (19) with parameters (α, β, f) is transformed in the wave with parameters $\left(\alpha', \beta', g \right)$ by the bi-filter

$$\left\{ \begin{array}{l} H_{22} = k H_{12} \\ H_{21} = k H_{11} \end{array} \right. \tag{20}$$

where the new parameters are (k, α, β are scalar, f and the H_{jk} are functions of ω)

$$\left\{ \begin{array}{l} \alpha' = \left(1 + |k|^2 \right)^{-1}, \qquad \beta' = k^* \left(1 + |k|^2 \right)^{-1} \\ |g|^2 = |f|^2 \left(1 + |k|^2 \right) \left((1 - \alpha) \left| H_{12} \right|^2 + \alpha \left| H_{11} \right|^2 + 2\beta \mathcal{R} \left[H_{12} H_{11}^* \right] \right). \end{array} \right.$$

4 About the Beer-Lambert law

4.1 The Beer-Lambert law for filters

In acoustics or ultrasonics, a beam is represented by a (real or complex) scalar quantity $U^z (t)$ parametrized by the distance z at a transmitter. Very often, the propagation medium is modelled by a LIF. If the medium can be split in "independent" pieces, the complex gain of the equivalent filter verifies the functional equation

$$H^{z_1 z_2} (\omega) = H^{z_1 u} (\omega) H^{u z_2} (\omega) \tag{21}$$

where $H^{z_1 z_2}(\omega)$ is the complex gain of the piece of medium in the interval (z_1, z_2). (21) expresses that the pieces (z_1, u) and (u, z_2) have the behavior of filters in series. Regular solutions of (21) on $\mathbb{R}^+$ are in the form [6]

$$H^{uv}(\omega) = e^{a(v,\omega)-a(u,\omega)}.$$

But if we suppose that

$$H^{uv}(\omega) = H^{v-u}(\omega)$$

(this means that only the thickness of the pieces appears) the last equality is reduced to

$$H^z(\omega) = e^{-za(\omega)} \tag{22}$$

where $a(\omega)$ depends only on the medium properties. For example, the model $a(\omega) = -\alpha\omega^2$ $(\alpha > 0)$ is admitted in limited frequency bands for atmosphere or water acoustic propagation, but other functions can be used [2], [13], [16], [8]. For optical propagation in free space, the function $a(\omega)$ is very complicated due to deep absorption holes. (22) is the Beer-Lambert law used in many domains of science. Finally, the spectral density $s^z(\omega)$ and the power P_z of the process $\mathbf{U}^z$ verify

$$s^z(\omega) = e^{-2z\mathcal{R}[a(\omega)]} s^0(\omega), \qquad P_z = \int_{-\infty}^{\infty} e^{-2z\mathcal{R}[a(\omega)]} s^0(\omega)\, d\omega.$$

4.2 The Beer-Lambert law for bi-filters

1) We ask the question to know if a generalization of the Beer-Lambert law can be done when bi-filters are used. Of course, we assume that the successive pieces of the medium do not interact and that their properties are independent of the beams which cross them. Each piece in the interval (u, v) is represented by four LIF of complex gains H_{jk}^{uv} with respect to the basis Oxy. We assume some geometric coherence of the medium so that

$$H_{jk}^{uv} = H_{jk}^{v-u}. \tag{23}$$

Figure 2 gives the equivalent circuit for two successive layers. To alleviate the formulae, we generally omit the variable ω. Consequently to the (strong) condition (23), and using the scheme of figure 2, the problem can be translated in a set of differential linear equations. We assume the existence of the derivatives $h_{jk}^z = \frac{\partial}{\partial z} H_{jk}^z$ for $z = 0$. We prove in appendix 4 (case 1) that, when λ_1, λ_2 are distinct different of 0 roots of

$$\lambda^2 - \left(h_{11}^0 + h_{22}^0\right)\lambda + h_{11}^0 h_{22}^0 - h_{12}^0 h_{21}^0 = 0 \tag{24}$$

the only one solution of the problem is

$$\left\{ \begin{array}{l} H_{11}^z = \frac{h_{11}^0 - \lambda_2}{\lambda_1 - \lambda_2} e^{z\lambda_1} + \frac{\lambda_1 - h_{11}^0}{\lambda_1 - \lambda_2} e^{z\lambda_2} \\ H_{12}^z = \frac{h_{12}^0}{\lambda_1 - \lambda_2}\left(e^{z\lambda_1} - e^{z\lambda_2}\right) \\ H_{21}^z = \frac{h_{21}^0}{\lambda_1 - \lambda_2}\left(e^{z\lambda_1} - e^{z\lambda_2}\right) \\ H_{22}^z = \frac{h_{22}^0 - \lambda_2}{\lambda_1 - \lambda_2} e^{z\lambda_1} + \frac{\lambda_1 - h_{22}^0}{\lambda_1 - \lambda_2} e^{z\lambda_2}. \end{array} \right. \tag{25}$$

Figure 2:

These equations are for bi-filters the version of the Beer-Lambert law for filters where the medium is represented by the family of complex gains $H^z(\omega) = e^{-za(\omega)}$. Also equations (25) define the set of admissible representations of a continuous medium with bi-filters in series. Firstly we study the case where the derivatives $h_{jk}^z = \frac{\partial}{\partial z} H_{jk}^z$ verify the conditions

$$h_{11}^0 = h_{22}^0, \quad h_{12}^0 = -h_{21}^0. \tag{26}$$

2) Let assume that h_{12}^0 does not depend on the frequency $\omega/2\pi$. We find that the complex gains H_{jk}^z verify

$$\begin{cases} H_{11}^z = H_{22}^z, & H_{12}^z = -H_{21}^z \\ H_{21}^z = e^{zh_{11}^0} \sin zh_{12}^0, & H_{11}^z = e^{zh_{11}^0} \cos zh_{12}^0. \end{cases} \tag{27}$$

If h_{12}^0 is a real quantity, we recognize the formulas for a rotation of the polarized beam (sections 3.6 and 6.2). The angle of rotation is equal to $-zh_{12}^0$ (proportional to z), and the amplitude $\mathbf{A}^z$ is the result of a LIF with input $\mathbf{A}^0$ and complex gain $e^{zh_{11}^0}$ (h_{11}^0 can depend on ω). Then the Beer-Lambert law (22) is true for the amplitude when the relations (26) are fulfilled for real h_{12}^0. Also h_{11}^0 is not real, because the imaginary part holds the propagation time between the points O and z (which cannot cancel and which depends on the frequency $\omega/2\pi$ in case of dispersion). The real part measures the weakening of the wave between both points.

To summarize, we can choose the parameters of the bi-filter to model a beam with a given polarization at 0, which is rotated by any angle, and with any weakening and with any propagation time. The (generally complex) amplitude obeys the usual Beer-Lambert law. The angle is ruled by h_{12}^0, the weakening by $\mathcal{R}\left[h_{11}^0\right]$, the phase by $\mathcal{I}\left[h_{11}^0\right]$, and the three parameters are proportional to the distance z.

3) Now we consider any beam with its components $E_x^0(t), E_y^0(t)$. When (26) is verified for the h_{jk}^0, these equalities are still true for the H_{jk}^z, and these quantities are invariant by rotation ($K_{jk}^z = H_{jk}^z$ using (9) and (10)). Then both components have the same behavior when crossing the medium, i.e. are rotated, delayed and weakened by same quantities. In the basis Ox'y' such that $-zh_{12}^0 =$ (Ox, Ox'), we have

$$E_{x'}^z(t) = \mathcal{F}\left[\mathbf{E}_x^0\right](t), \quad E_{y'}^z(t) = \mathcal{F}\left[\mathbf{E}_y^0\right](t)$$
$$F(\omega) = e^{zh_{11}^0(\omega)}$$

where $F(\omega)$ is the complex gain of the LIF $\mathcal{F}$. Consequently the power at z will be given by (from (8) and the Wiener-Lee relations, see appendix 1)

$$P_z = \int_{-\infty}^{\infty} e^{2z\mathcal{R}\left[h_{11}^0(\omega)\right]} \left[s_x^0 + s_y^0\right](\omega)\, d\omega \tag{28}$$

to be compared with the result in the standard case (see the end of section 4.1). Then the power P_z at z is independent of the rotation defined by the parameter h^0_{12}. Equivalently, when studying the power, the beam state of polarization is irrelevant, and the medium can be viewed as a simple LIF where the Beer-Lambert law is available with the parameter h^0_{11}.

However, if we leave off the hypothesis of independence of h^0_{12} with the frequency, the interpretation of the bi-filter as a rotator is not true because the angle of rotation is now a quantity proper to each frequency.

4) When $h^0_{12} = \rho e^{i\phi}$ is no longer real (but still independent of the frequency), (27) is true but with complex trigonometric functions which are expanded in

$$\left\{\begin{array}{c} H^z_{11} = a\cos\left(z\rho\cos\phi\right) + b\cos\left(z\rho\cos\phi - \frac{\pi}{2}\right) \\ H^z_{12} = a\sin\left(z\rho\cos\phi\right) + b\sin\left(z\rho\cos\phi - \frac{\pi}{2}\right) \\ a = e^{zh^0_{11}}\cosh\left(z\rho\sin\phi\right) \\ b = -ie^{zh^0_{11}}\sinh\left(z\rho\sin\phi\right). \end{array}\right.$$

This means that a polarized wave is split in two parts, the first one rotated by the angle $-z\rho$ cosϕ, with amplitude coming from a LIF $\mathcal{A}$ with complex gain a, the second one rotated by $\left(-z\rho cos\phi + \frac{\pi}{2}\right)$ and with the complex gain b of a LIF $\mathcal{B}$. The power P_z is given by (the calculus is performed in Ox'y' with $-z\rho cos\phi =$(Ox, Ox'))

$$P_z = \mathrm{E}\left\{\left[\left|\mathcal{A}\left[\mathbf{E}^0_x\right] - \mathcal{B}\left[\mathbf{E}^0_y\right]\right|^2 + \left|\mathcal{B}\left[\mathbf{E}^0_x\right] + \mathcal{A}\left[\mathbf{E}^0_y\right]\right|^2\right](t)\right\}$$

Computations can be performed (to simplify we give results for an unpolarized wave $s^0_{xy} = 0$)

$$P_z = \int_{-\infty}^{\infty}\left[e^{2z\mathcal{R}\left[h^0_{11}\right]}\left(s^0_x + s^0_y\right)\cosh\left(2\rho z\sin\phi\right)\right](\omega)\,d\omega.$$

We see that (28) is no longer true.

Assume that the wave is polarized at O. Two reasons lead to a depolarization of the wave. The first one is the dependency of the parameter h^0_{12} on frequency. Then the angle of rotation of the beam is different following the frequency. The second one happens when h^0_{12} is not real, which creates a secondary wave with amplitude b and orthogonal to the main wave (of amplitude a). Because the function tanh is increasing on $\mathbb{R}$, we have the same property for the quotient $|b/a|$ (but which cannot reach 1).

5) We go back to the general formulas (25). They show that the beam can be split in two parts, the first one containing the terms with coefficient $e^{z\lambda_1}$ and the second part with $e^{z\lambda_2}$. Generally λ_1 and λ_2 are not conjugate complex numbers because the equation (24) can have complex coefficients. Assume that the input is the pure wave $e^{i\omega_0 t}$. If

$$\lambda_1 = a + ib, \quad \lambda_2 = c + id$$

the wave at z will be the sum of two pure waves which cross the medium with celerities $-\omega_0/b$ for the first one and $-\omega_0/d$ for the second one and with attenuation ruled by a and c. Celerities are different when $b \neq d$ as in a birefringent medium. Obviously, each wave obeys the Beer-Lambert law in its simplest form but not the sum, except when the h_{jk} verify some conditions. Finally, equations (25) are given when the roots of (24) are distinct and different of 0. Other situations are described in the appendix 4.

5 Conclusion

A bi-filter is defined by a circuit of four linear invariant filters. It is a particular case 2x2 of the MIMO circuits (for multiple inputs-multiple outputs) which are used for instance in communications between systems of antennas [14] and in sampling to improve the reconstruction of signals [17]. It generalizes the well-known notion of "scattering matrix" in radar processing. In this paper, we address the problem of modelling a medium crossed by a beam with two components and with some degree of polarization. We assume that the beams are random processes with properties of stationarity and spectra with any bandwith. We prove that bi-filters explain elementary operations on electromagnetic waves and we establish a generalization of the Beer-Lambert law which justifies the model in continuous media. Obviously, other situations can be highlighted. The parameters of the bi-filter are defined by the physical properties of the medium. Perhaps theoretical considerations about the crossed material could allow the determination of these parameters but I feel that it is a difficult task. In the field of ultrasonics, for instance, the attenuation and the celerity of waves in some medium (sea water, biological tissues...) are obtained by experiments and not from the mechanical, chemical....considerations. I believe that the same applies for electromagnetic waves. Characteristics of a material are measured by studying a set of waves with different frequency and polarization. These experiments are able to estimate the four complex gains which define a bi-filter and they can help to give a fair representation of the medium.

6 Appendices

6.1 Appendix 1: Bi-filter

1) Firstly, we summarize the usual theory of linear invariant filtering (LIF) of stationary processes [1], [12], [4]. If $\mathbf{X} = \{X(t), t \in \mathbb{R}\}$ is characterized by its spectral density $s_X(\omega)$, it is possible to define an isometry I_X between the Hilbert spaces $\mathbf{H}_X$ and $\mathbf{K}_s$ where

-$\mathbf{H}_X$ is the set of linear combinations of the random variables $X(t)$

(completed by the closure of this set) when the scalar product $\langle .,. \rangle_H$ and the associated distance d are used:

$$\begin{cases} \langle X(u), X(v) \rangle_{\mathbf{H}} = \mathrm{E}[X(u) X^*(v)] \\ d^2(A,B) = \mathrm{E}\left[|A-B|^2\right] \end{cases}$$

-$\mathbf{K}_s$ is the set of the $f(\omega)$ (from $\mathbb{R}$ to $\mathbb{C}$) such as $\int_{-\infty}^{\infty} |f(\omega)|^2 s_X(\omega)\, d\omega < \infty$, with the scalar product $\langle .,. \rangle_K$ and distances defined by

$$\begin{cases} \langle f, g \rangle_{\mathbf{K}} = \int_{-\infty}^{\infty} [f g^* s_X](\omega)\, d\omega \\ d^2(f,g) = \int_{-\infty}^{\infty} \left[|f-g|^2 s_X\right](\omega)\, d\omega \end{cases}.$$

The isometry I_X is defined by the relation

$$X(t) \longleftrightarrow_{I_X} e^{i\omega t} \tag{29}$$

The isometry maintains the scalar product and the distances of corresponding elements of the spaces. Consequently, it allows to perform calculations in the space $\mathbf{K}_s$ rather than in $\mathbf{H}_X$, using Fourier analysis and geometry of Hilbert spaces.

If $\mathcal{F}$ and $\mathcal{G}$ are two LIF with complex gains $F(\omega)$ and $G(\omega)$, input $\mathbf{X}$, outputs $\mathbf{U}$=$\mathcal{F}[\mathbf{X}]$ and $\mathbf{V}$=$\mathcal{G}[\mathbf{X}]$, we have

$$\begin{cases} U(t) \longleftrightarrow_{I_X} F(\omega) e^{i\omega t}, \quad V(t) \longleftrightarrow_{I_X} G(\omega) e^{i\omega t} \\ \mathrm{E}[U(t) V^*(t-\tau)] = \int_{-\infty}^{\infty} e^{i\omega\tau} [F G^* s_X](\omega)\, d\omega \\ s_U(\omega) = \left[|F|^2 s_X\right](\omega), \quad s_V(\omega) = \left[|G|^2 s_X\right](\omega) \\ s_{UV}(\omega) = [F G^* s_X](\omega) \end{cases} \tag{30}$$

where $s_U(\omega), s_V(\omega), s_{UV}(\omega)$ are spectral and cross-spectral densities.

Finally, if $\mathbf{W}$=$\mathcal{H}[\mathbf{U}]$, and if $H(\omega)$ is the complex gain of the LIF $\mathcal{H}$, we have

$$W(t) \longleftrightarrow_{I_X} [HF](\omega) e^{i\omega t}$$

which is the relation for filters in series.

2) In section 2, the processes $\mathbf{X}_1$ and $\mathbf{Y}_1$ are stationary and stationary correlated. If we look at processes $\mathbf{Y}_1'$ and $\mathbf{Y}_1''$ defined by

$$\begin{cases} Y_1(t) = Y_1'(t) + Y_1''(t) \\ Y_1'(t) \longleftrightarrow_{I_{X_1}} \left[\frac{s_{Y_1 X_1}}{s_{X_1}}\right](\omega) e^{i\omega t} \end{cases} \tag{31}$$

$\mathbf{X}_1$ and $\mathbf{Y}_1'$ are the input and the output of a LIF filter of complex gain $s_{Y_1X_1}/s_{X_1}$. Using (30) we obtain, whatever $t, \tau \in \mathbb{R}$

$$\mathrm{E}\left[X_1(t) Y_1''^*(t-\tau)\right] = 0.$$

This means that $Y_1'(t)$ is the orthogonal projection of $Y_1(t)$ on $\mathbf{H}_{X_1}$ and that $Y_1''(t)$ is orthogonal to $\mathbf{H}_{X_1}$, and

$$\begin{cases} Y_1'(t) \in \mathbf{H}_{X_1} \qquad Y_1''(t) \perp \mathbf{H}_{X_1} \\ s_{Y_1'} = \frac{\left|s_{Y_1X_1}\right|^2}{s_{X_1}}, \quad s_{X_1Y_1'} = s_{X_1Y_1} \\ s_{Y_1''} = s_{Y_1} - \frac{\left|s_{Y_1X_1}\right|^2}{s_{X_1}} \end{cases} \tag{32}$$

where $s_{Y_1'}, s_{X_1Y_1'}, s_{Y_1''}$... are spectral and cross-spectral densities. Consequently, (4) can be split in two orthogonal systems $\mathbf{S}'$ and $\mathbf{S}''$

$$\mathbf{S}' \begin{cases} X_2'(t) = \mathcal{H}_{11}[\mathbf{X}_1](t) + \mathcal{H}_{12}[\mathbf{Y}_1'](t) \\ Y_2'(t) = \mathcal{H}_{21}[\mathbf{X}_1](t) + \mathcal{H}_{22}[\mathbf{Y}_1'](t) \end{cases} \tag{33}$$

$$\mathbf{S}'' \begin{cases} X_2''(t) = \mathcal{H}_{12}[\mathbf{Y}_1''](t) \\ Y_2''(t) = \mathcal{H}_{22}[\mathbf{Y}_1''](t) \end{cases}$$

$$\begin{aligned} Y_1(t) &= Y_1'(t) + Y_1''(t) \\ Y_2(t) &= Y_2'(t) + Y_2''(t) \\ X_2(t) &= X_2'(t) + X_2''(t) \end{aligned}$$

By construction, the sets $\mathbf{S}' = (\mathbf{X}_2', \mathbf{Y}_2')$ and $\mathbf{S}'' = (\mathbf{X}_2'', \mathbf{Y}_2'')$ are uncorrelated. Each equation of $\mathbf{S}'$ is equivalent to a circuit composed by three filters. $\mathbf{X}_2'$ is the output of the LIF with complex gain $H_{11} + \frac{s_{Y_1X_1}}{s_{X_1}}H_{12}$ with input $\mathbf{X}_1$. The filter of complex gain $H_{21} + \frac{s_{Y_1X_1}}{s_{X_1}}H_{22}$ is for $\mathbf{Y}_2'$.

3) From (30), (32), (33) we deduce the spectral characteristics of $(\mathbf{Y}_1, \mathbf{Y}_2)$ (we omit the variable ω)

$$\begin{cases} s_{X_2Y_2} = \left(\frac{s_{Y_1X_1}}{s_{X_1}}H_{12} + H_{11}\right)\left(\frac{s_{Y_1X_1}}{s_{X_1}}H_{22} + H_{21}\right)^* s_{X_1} + H_{12}H_{22}^*\left[s_{Y_2} - \frac{\left|s_{Y_1X_1}\right|^2}{s_{X_1}}\right] \\ s_{X_2} = \left|\frac{s_{Y_1X_1}}{s_{X_1}}H_{12} + H_{11}\right|^2 s_{X_1} + \left|H_{12}\right|^2\left[s_{Y_1} - \frac{\left|s_{Y_1X_1}\right|^2}{s_{X_1}}\right] \\ s_{Y_2} = \left|\frac{s_{Y_1X_1}}{s_{X_1}}H_{22} + H_{21}\right|^2 s_{X_1} + \left|H_{22}\right|^2\left[s_{Y_1} - \frac{\left|s_{Y_1X_1}\right|^2}{s_{X_1}}\right] \\ s_{X_1X_2} = \left(\frac{s_{Y_1X_1}}{s_{X_1}}H_{12} + H_{11}\right)^* s_{X_1} \\ s_{X_1Y_2} = \left(\frac{s_{Y_1X_1}}{s_{X_1}}H_{22} + H_{21}\right)^* s_{X_1} \\ s_{Y_1X_2} = \frac{s_{Y_1X_1}}{s_{X_1}}\left(\frac{s_{Y_1X_1}}{s_{X_1}}H_{12} + H_{11}\right)^* s_{X_1} + H_{12}^*\left[s_{Y_1} - \frac{\left|s_{Y_1X_1}\right|^2}{s_{X_1}}\right] \\ s_{Y_1Y_2} = \frac{s_{Y_1X_1}}{s_{X_1}}\left(\frac{s_{Y_1X_1}}{s_{X_1}}H_{22} + H_{21}\right)^* s_{X_1} + H_{22}^*\left[s_{Y_1} - \frac{\left|s_{Y_1X_1}\right|^2}{s_{X_1}}\right] \end{cases} \tag{34}$$

6.2 Appendix 2: Rotator

1) We study bi-filters which have polarized beams as input and output. We look for the bi-filters which increase the angle of polarization by a given

quantity θ, independently of the absolute values of the angle at the input and independently of the power spectra.

Assume that the beam is polarized at 0 and z with angles ψ and $\psi' = \psi + \theta$ (with respect to Oxy). It is equivalent to have (section 3-1)

$$\begin{cases} A^z(t)\cos\psi' = [\mathcal{H}_{11}\cos\psi + \mathcal{H}_{12}\sin\psi]\left[\mathbf{A}^0\right](t) \\ A^z(t)\sin\psi' = [\mathcal{H}_{21}\cos\psi + \mathcal{H}_{22}\sin\psi]\left[\mathbf{A}^0\right](t) \end{cases}$$

Taking $\psi = 0$ and $\psi = \pi/2$ leads to

$$\begin{cases} \mathcal{H}_{21}\left[\mathbf{A}^0\right] = \mathcal{H}_{11}\left[\mathbf{A}^0\right]\tan\theta \\ \mathcal{H}_{22}\left[\mathbf{A}^0\right] = -\mathcal{H}_{12}\left[\mathbf{A}^0\right]\cot\theta \end{cases}$$

equalities which are true for any $\mathbf{A}^0$. Then the equalities about the complex gains become

$$\begin{cases} H_{21}\cos\theta = H_{11}\sin\theta \\ H_{12}\cos\theta = -H_{22}\sin\theta \end{cases} \tag{35}$$

We enter (35) in the first equality which becomes

$$\begin{cases} A^z(t)\cos(\psi+\theta) = [\mathcal{H}_{11}\cos\psi - \mathcal{H}_{22}\sin\psi\tan\theta]\left[\mathbf{A}^0\right](t) \\ A^z(t)\sin(\psi+\theta) = [\mathcal{H}_{11}\cos\psi\tan\theta + \mathcal{H}_{22}\sin\psi]\left[\mathbf{A}^0\right](t) \end{cases}$$

and we deduce the equality

$$\cot(\psi+\theta) = \frac{H_{11}\cos\psi\cos\theta - H_{22}\sin\psi\sin\theta}{H_{11}\cos\psi\sin\theta + H_{22}\sin\psi\cos\theta}$$

which has to be true whatever ψ. Obviously, it is possible if and only if $H_{11} = H_{22}$. Using (35) , we conclude that a NSC for a bi-filter to induce a rotation of angle θ is summarized by

$$\begin{cases} H_{21} = -H_{12} = H_{11}\tan\theta \\ H_{22} = H_{11} \\ A^z(t) = \frac{1}{\cos\theta}\mathcal{H}_{11}\left[\mathbf{A}^0\right](t) \end{cases} \tag{36}$$

except $\theta \neq \frac{\pi}{2} \bmod \pi$. For this particular case, we have

$$\begin{cases} H_{11} = H_{22} = 0, \quad H_{12} = -H_{21} \\ A^z(t) = -\mathcal{H}_{12}\left[\mathbf{A}^0\right](t)\,. \end{cases}$$

To summarize, bi-filters verifying (36) define transformations composed by a rotation of the direction of polarization, associated to a LIF for the amplitude. The LIF has input $\mathbf{A}^0$, output $\mathbf{A}^z$ and complex gain $H_{11}/\cos\theta$ (for $\theta \neq \frac{\pi}{2} \bmod \pi$).

Then a "pure rotation" which retains the amplitude corresponds to the bi-filter

$$\begin{cases} H_{11} = H_{22} = \cos\theta \\ H_{21} = -H_{12} = \sin\theta \end{cases}$$

which is not surprising.

6.3 Appendix 3: Unpolarized wave

1) If $s_{xy}^{z}=0$ whatever the system of coordinates, we have, when $\theta=(\mathrm{Ox},\mathrm{Ox}')$

$$\left\{\begin{array}{l} s_{x'y'}^{z}=\left(s_{y}^{z}-s_{x}^{z}\right)\sin\theta\cos\theta=0 \\ s_{x'}^{z}=s_{x}^{z}\cos^{2}\theta+s_{y}^{z}\sin^{2}\theta \\ s_{y'}^{z}=s_{y}^{z}\cos^{2}\theta+s_{x}^{z}\sin^{2}\theta \end{array}\right. \tag{37}$$

whatever θ, and then

$$s_{x'}^{z}=s_{y'}^{z}=s_{x}^{z}=s_{y}^{z}.$$

The definition which we have taken for the unpolarized beam holds on the whole process $\overrightarrow{\mathbf{E}^{z}}$ and not only on the two-dimensional random variable $\left(E_{x}^{z}(t),E_{y}^{z}(t)\right)$. It takes into account all $\left(E_{x}^{z}(t),E_{y}^{z}(t')\right)$. It is a strong difference.

As an example, take $E_{y}^{z}(t)=\mathcal{H}\left[\mathbf{E}_{x}^{z}\right](t)$, where $\mathcal{H}[..]$ is the Hilbert tranform and $\mathbf{E}_{x}^{z}$ is a real process. It is wellknown that this implies $\rho_{xy}^{z}=0,\rho_{x}^{z}=\rho_{y}^{z}$ and these equalities remain true in any coordinates systems (the equations (37) are verified for variances and covariances). Also

$$\left\{\begin{array}{l} s_{x}^{z}=s_{y}^{z}=s_{x'}^{z}=s_{y'}^{z} \\ s_{xy}^{z}=s_{x'y'}^{z}=-is_{x}^{z}\mathrm{sign} \end{array}\right.$$

where $\mathrm{sign}\omega=1$ for $\omega>0$ and -1 for $\omega<0$. Though $\rho_{xy}^{z}=0$ in any system ($s_{x}^{z}(\omega)$ is even), the r.v. $E_{x}^{z}(t)$ and $E_{y}^{z}(t')$ are linked for different t,t' because the cross-spectrum is different of 0.

2) A compensator corresponds to a bi-filter such that (in Oxy)

$$\mathbf{H}=\left[\begin{array}{cc} e^{-i\omega\theta_{x}} & 0 \\ 0 & e^{-i\omega\theta_{y}} \end{array}\right]$$

θ_{x} and θ_{y} are the delays applied to E_{x}^{0} and E_{y}^{0}. If we assume $s_{xy}^{0}=0$, the first formula of (7) implies that $s_{xy}^{z}=0$. Then a compensator maintains the property of unpolarization when the strong definition is used.

If we take $\mathbf{E}_{x}^{0}=\mathbf{A}+\mathbf{B},\mathbf{E}_{\mathbf{y}}^{\mathbf{0}}=\mathbf{A}-\mathbf{B}$ where $\mathbf{B}$ is the Hilbert tranform of $\mathbf{A}$ with real $\mathbf{A}$, we obtain ($\mathrm{sgn}\omega=1$ for $\omega>0$ and -1 for $\omega<0$)

$$s_{x}^{0}=s_{y}^{0}=s_{x}^{z}=s_{y}^{z}=2s_{A},s_{xy}^{0}=-2is_{A}\mathrm{sgn}$$

which implies $\rho_{xy}^{0}=0$ because $s_{A}^{0}(\omega)$ is even. Then, the beam is unpolarized at O in the weak sense but not in the strong sense. However from (7)

$$\left\{\begin{array}{l} s_{xy}^{z}(\omega)=-2ie^{i\omega(\theta_{y}-\theta_{x})}s_{A}(\omega)\,\mathrm{sgn}\omega \\ \rho_{xy}^{z}=4\int_{0}^{\infty}s_{A}(\omega)\sin\omega\left(\theta_{y}-\theta_{x}\right)d\omega. \end{array}\right.$$

Obviously we do not generally have $\rho_{xy}^{z}=0$, which proves that a compensator does not maintain the unpolarization in the weak definition, except if the spectral support of $s_{A}(\omega)$ is small enough around some ω_{0} and $\omega_{0}\left(\theta_{y}-\theta_{x}\right)$ close to a multiple of π.

From (7) and with the strong definition, the unpolarization is maintained from O to z if and only if

$$H_{12}H_{22}^{*}+H_{11}H_{21}^{*}=0.$$

6.4 Appendix 4: the Beer-Lambert law

1) The beam state $\overrightarrow{\mathbf{E}^{z}}=\left(\mathbf{E}_{x}^{z},\mathbf{E}_{y}^{z}\right)$ at z is the result of the bi-filtering of $\overrightarrow{\mathbf{E}^{0}}$ by the H_{jk}^{0z} or the bi-filtering of $\overrightarrow{\mathbf{E}^{u}}, u<z$, by the H_{jk}^{uz} (of course they are function of ω but we can omit this variable). This leads to the equations (using elementary properties of circuits and looking at figure 2)

$$\left\{\begin{array}{l} H_{11}^{0z}=H_{11}^{0u}H_{11}^{uz}+H_{21}^{0u}H_{12}^{uz} \\ H_{12}^{0z}=H_{12}^{0u}H_{11}^{uz}+H_{22}^{0u}H_{12}^{uz} \\ H_{21}^{0z}=H_{11}^{0u}H_{21}^{uz}+H_{21}^{0u}H_{22}^{uz} \\ H_{22}^{0z}=H_{12}^{0u}H_{21}^{uz}+H_{22}^{0u}H_{22}^{uz} \end{array}\right.$$

which are simplified in (from (23) which translates the homogeneity of the medium)

$$\left\{\begin{array}{l} H_{11}^{z}=H_{11}^{u}H_{11}^{z-u}+H_{21}^{u}H_{12}^{z-u} \\ H_{12}^{z}=H_{12}^{u}H_{11}^{z-u}+H_{22}^{u}H_{12}^{z-u} \\ H_{21}^{z}=H_{11}^{u}H_{21}^{z-u}+H_{21}^{u}H_{22}^{z-u} \\ H_{22}^{z}=H_{12}^{u}H_{21}^{z-u}+H_{22}^{u}H_{22}^{z-u} \end{array}\right. \tag{38}$$

For instance , we write the first equation under the form

$$\frac{H_{11}^{z+a}-H_{11}^{z}}{a}=H_{11}^{z}\frac{H_{11}^{a}-1}{a}+H_{21}^{z}\frac{H_{12}^{a}}{a}.$$

If we assume the existence of derivatives $h_{jk}^{z}=\frac{\partial}{\partial z}H_{jk}^{z}$ we obtain

$$\left\{\begin{array}{l} h_{11}^{z}=H_{11}^{z}h_{11}^{0}+H_{21}^{z}h_{12}^{0} \\ h_{12}^{z}=H_{12}^{z}h_{11}^{0}+H_{22}^{z}h_{12}^{0} \\ h_{21}^{z}=H_{11}^{z}h_{21}^{0}+H_{21}^{z}h_{22}^{0} \\ h_{22}^{z}=H_{12}^{z}h_{21}^{0}+H_{22}^{z}h_{22}^{0} \\ \text{with}\quad h_{jk}^{z}=\frac{d}{dz}H_{jk}^{z} \end{array}\right. \tag{39}$$

which includes the (realistic) conditions

$$\lim_{z\to 0}H_{11}^{z}=\lim_{z\to 0}H_{22}^{z}=1 \quad \text{and} \quad \lim_{z\to 0}H_{12}^{z}=\lim_{z\to 0}H_{21}^{z}=0. \tag{40}$$

The system can be written as the matricial equation

$$\mathbf{h}^{u}=\mathbf{P}\mathbf{H}^{u},\quad \mathbf{P}=\left[\begin{array}{cccc} h_{11}^{0} & 0 & h_{12}^{0} & 0 \\ 0 & h_{11}^{0} & 0 & h_{12}^{0} \\ h_{21}^{0} & 0 & h_{22}^{0} & 0 \\ 0 & h_{21}^{0} & 0 & h_{22}^{0} \end{array}\right] \tag{41}$$

Three cases can be highlighted, following the properties of $\mathbf{P}$. We assume that $H^{\infty}_{jk} = 0$ because a wave in a unlimited medium is evanescent. This condition cancels constants which can appear in solutions of the system.

2) We have three possibilities which are detailed below.

$$\text{Case 1 :} \left\{ \begin{array}{c} \left(h^0_{11} - h^0_{22}\right)^2 + 4h^0_{12}h^0_{21} \neq 0 \text{ and } h^0_{11}h^0_{22} \neq h^0_{12}h^0_{21} \\ H^z_{jk} = c_{jk1}e^{\lambda_1 z} + c_{jk2}e^{\lambda_2 z} \end{array} \right.$$

where λ_1, λ_2 are distinct eigenvalues of $\mathbf{P}$ (which have negative real parts for a passive medium). The conditions (40) lead to

$$\left\{ \begin{array}{l} H^z_{11} = d_{11}e^{z\lambda_1} + (1 - d_{11})\, e^{z\lambda_2} \\ H^z_{12} = d_{12}e^{z\lambda_1} - d_{12}e^{z\lambda_2} \\ H^z_{21} = d_{21}e^{z\lambda_1} - d_{21}e^{z\lambda_2} \\ H^z_{22} = d_{22}e^{z\lambda_1} + (1 - d_{22})\, e^{z\lambda_2} \end{array} \right. \tag{42}$$

with, using (39)

$$\begin{array}{cc} d_{11} = \frac{h^0_{11} - \lambda_2}{\lambda_1 - \lambda_2}, & d_{12} = \frac{h^0_{12}}{\lambda_1 - \lambda_2} \\ d_{21} = \frac{h^0_{21}}{\lambda_1 - \lambda_2}, & d_{22} = \frac{h^0_{22} - \lambda_2}{\lambda_1 - \lambda_2} \end{array} \tag{43}$$

where λ_1 and λ_2 are (distinct, different of 0 and with negative real parts) roots of

$$\lambda^2 - \left(h^0_{11} + h^0_{22}\right)\lambda + h^0_{11}h^0_{22} - h^0_{12}h^0_{21} = 0 \tag{44}$$

Conversely, (42) with (43) verify (38). Moreover formulae (11) imply the invariance of λ_1 and λ_2 in any rotation. We obtain the set of K^z_{jk} fitted to Ox'y' by replacing h^0_{jk} by $k^0_{jk} = \frac{\partial}{\partial z}K^0_{jk}$ in (42), (43). This case is developped in section 4.2.

$$\text{Case 2 :} \left\{ \begin{array}{c} \left(h^0_{11} - h^0_{22}\right)^2 + 4h^0_{12}h^0_{21} \neq 0 \text{ and } h^0_{11}h^0_{22} = h^0_{12}h^0_{21} \\ H^z_{jk} = c_{jk}e^{\lambda z} \end{array} \right.$$

where $\lambda = h^0_{11} + h^0_{22}$ is the eigenvelue of $\mathbf{P}$ assumed different of 0. Conditions (40) imply

$$c_{11} = c_{22} = 1, \quad c_{12} = c_{21} = 0.$$

Obviously the usual Beer-Lambert law is verified. Each component is weakened and delayed through a quantity proportional to z.

$$\text{Case 3 :} \left(h^0_{11} - h^0_{22}\right)^2 + 4h^0_{12}h^0_{21} = 0 \; .$$

In the last case, $\mathbf{P}$ has only one eigenvalue (of order 4) wich corresponds to a proper subspace of dimension 2. Actually, we find same results as in the case 2.

Then we have shown that the system (38) has an unique solution most of the time depending of two parameters (for instance h_{11}^0, h_{12}^0). However they may be functions of the frequency $\omega/2\pi$. Consequently bi-filters are able to model propagation of electromagnetic beams through continuous (and stationary) media such as free space or optical fiber or coaxial cable.

6.5 Appendix 5: a class of beams

The simplest model of incoherent light is described by

$$\begin{cases} X(t) = \sum_j e^{i\omega_0(t-t_j)} h(t-t_j)\cos\Theta_j \\ Y(t) = \sum_j e^{i\omega_0(t-t_j)} h(t-t_j)\sin\Theta_j \end{cases}$$

where $\mathbf{t}$=$\{t_n, n \in \mathbb{Z}\}$ is an homogeneous Poisson process with parameter λ, the Θ_n are random variables independent of $\mathbf{t}$ and between them, and $h(t)$ is regular enough. Each term represents the emission by the particle j at the time t_j in the direction Θ_j. Straigthforward calculations yield (assuming $H(\omega_0) = 0$ to suppress some continuous component)

$$\begin{cases} s_X(\omega) = \frac{1}{4\pi}|H(\omega-\omega_0)|^2 \mathrm{E}\left[\cos^2\Theta\right] \\ s_Y(\omega) = \frac{1}{4\pi}|H(\omega-\omega_0)|^2 \mathrm{E}\left[\sin^2\Theta\right] \\ s_{XY}(\omega) = \frac{1}{4\pi}|H(\omega-\omega_0)|^2 \mathrm{E}\left[\sin\Theta\cos\Theta\right] \\ H(\omega) = \int_{-\infty}^{\infty} h(u)\, e^{-i\omega u} du \end{cases}$$

Taking Θ_n uniformly distributed leads to (this means that the elementary emitters have no favourite polarization)

$$\begin{cases} s_X(\omega) = s_Y(\omega) = \frac{1}{4\pi}|H(\omega-\omega_0)|^2 \\ s_{XY}(\omega) = 0 \end{cases}$$

Then, the wave with components $\mathbf{X}, \mathbf{Y}$ is unpolarized.

When $\Pr[\Theta = \theta] = 1$, we have a polarized beam in the direction θ :

$$\begin{cases} s_X(\omega) = \frac{1}{4\pi}|H(\omega-\omega_0)|^2 \cos^2\theta \\ s_Y(\omega) = \frac{1}{4\pi}|H(\omega-\omega_0)|^2 \sin^2\theta \\ s_{XY}(\omega) = \frac{1}{4\pi}|H(\omega-\omega_0)|^2 \sin\theta\cos\theta \end{cases}$$

Other laws for Θ give a large choice of situations in the form (for real α, β)

$$\begin{cases} s_X(\omega) = \alpha\,|f(\omega)|^2 \\ s_Y(\omega) = (1-\alpha)\,|f(\omega)|^2 \\ s_{XY}(\omega) = \beta\,|f(\omega)|^2 \end{cases}$$

where $|\beta| \leq \sqrt{\alpha(1-\alpha)}$. Conversely, we can find a probability law for each value of (α, β).

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