Shape Effects on Sampling of Stationary Processes

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***Abstract*—Acquisition devices play an important role in digital signal processing. The possibility of a perfect reconstruction is demonstrated in regular as well as irregular sampling when the number of samples in the observation interval is high enough in function of the bandwidth of the sampled signal (length of the support of the spectrum). In the case of high sampling rates, imperfections of acquisition devices can introduce non negligible errors (when the acquisition duration of a given sample becomes not negligible in comparison with the sampling period (or mean sampling period in the case of irregular sampling). In this paper, explicit method is proposed to take into account imperfections of the sampling device in order to improve the reconstruction of the signal. The proposed method is applicable for deterministic functions and random processes in the case of regular sampling, as well as irregular sampling.**

I. Introduction

In 1897, E. Borel wrote that the function $f(z)$ defined as

$$f(z) = \int_{\Delta} \phi(x)\, e^{izx} dx \tag{1}$$

for regular enough $\phi(x), \Delta = [-\pi, \pi]$, is perfectly defined by the $f(\pm 1)$, $f(\pm 2)$, ... i.e by the value of f at a periodic sequence of points (exact assumptions can be found in [1]). The property can be extended to other kinds of sets Δ when some conditions are fulfilled (alias-free sets). More generally, Borel linked the theory of zeros of entire functions and interpolation which generalized the property to irregular sequences $f(t_n), n \in \mathbb{Z}$ where the t_n sequence is no longer periodic [2].

Actually, a realistic acquisition device has a working time different from zero. This means that resulting estimated samples of $f(t)$ are deduced from the behavior around time t and not from a local value at a single point. Numerous situations are studied in the literature: irregular or regular sampling, nonideal acquisition and random acquisition. When acquisition device effects are represented by a convolution product (between the signal to be sampled and the device's impulse response), solutions are proposed in [3]. The notion of consistent sampling was popularized in this last paper and it lightens constraints about spectral supports [4]. But those solutions are valid in the periodic case only (regular sampling). In this paper we present a solution for the non periodic sampling case and which is valid in the case of deterministic functions as well as random processes.

For stationary random processes, the S. P. Lloyd paper [5] can be viewed as the foundation of the sampling theory. In the periodic case, it generalizes the notion of bandwidth and the notion of Landau bound rather than Nyquist bound [6]. It links conditions of errorless reconstruction to the notion of alias-free spectra and proposes explicit formulas. Let us put in the framework of stationary random processes with acquisition by linear invariant filters around an irregular sequence of points. Errorless reconstruction depends on the spectral support and formulas are common to the set of processes with the same spectral support. A formula fitted for a given process is also true for its filtered versions. This property allows us to construct errorless formulas linking a random process realization and irregular samples coming from a nonideal acquisition device.

Explicit formula allowing errorless reconstruction using Periodic Nonuniform Sampling of order $2L$ (PNS$2L$) have already been published in [7]. In this paper, we demonstrate how they can be corrected to take into account the knowledge of the impulse response of a non ideal acquisition device. Section II presents mathematical problem formulation and simulations are carried out in section III, demonstrating the accuracy improvement of reconstruction. Section IV concludes the paper.

II. Problem formulation

A. Hypotheses

We consider a zero mean stationary process $\mathbf{Z} = \{Z(t), t \in \mathbb{R}\}$ with regular power spectral density $s_Z(\omega)$ defined by [8]

$$\mathrm{E}[Z(t)\, Z^*(t-\tau)] = \int_{\Delta} e^{i\omega\tau} s_Z(\omega)\, d\omega \tag{2}$$

where E[.] and the superscript * stand respectively for the mathematical expectation (or ensemble mean) and complex conjugate. Δ is the total spectral support of $\mathbf{Z}$:

$$\Delta = \{\omega \in \mathbb{R}, s_Z(\omega) > 0\}. \tag{3}$$

Moreover, we consider

- a sampling sequence $\mathbf{t} = \{t_n, n \in \mathbb{Z}\}$ defined as an increasing series of real numbers (sampling instants).
- some regular enough function $g(t)$ (impulse response of the acquisition device) defined on $\mathbb{R}$ with Fourier transform

$$G(\omega) = \int_{-\infty}^{\infty} g(t) e^{-i\omega t} dt. \tag{4}$$

Practically, $g(t)$ will be a function with a maximum nearby the origin $t = 0$, and quickly vanishing.

Let the process $\mathbf{U} = \{U(t), t \in \mathbb{R}\}$ be defined as

$$U(t) = \int_{-\infty}^{\infty} g(t-u) Z(u) du. \tag{5}$$

$\mathbf{U} = \mathcal{G}[\mathbf{Z}]$ is the output of a LIF (Linear Invariant Filter) with input $\mathbf{Z}$, complex gain $G(\omega)$ and impulse response $g(t)$. We assume that a finite sequence of data $U(t_n)$ is available at the output of the acquisition device. This means that the device provides a measurement of $Z(t)$ around the point $t = t_n$ approaching $Z(t_n)$ as $g(t)$ approaches a very narrow function with unit surface (Dirac function). We don't know the values of $Z(t)$ at determined points t_n but we know the behavior of $Z(t)$ in a neighborhood of t_n. This is a realistic view of sampling, taking into account the device producing the data. Formula (5) defines a "linear invariant acquisition", more general that a punctual one.

B. Interpolation taking into account the "shape" of the sampling device

In this section, we consider that the sampling device provides samples at known time instants (periodic or not) and with imperfect (but known) shape $g(t)$: the process $\mathbf{U}$ (representing the observed process, after imperfect sampling device) is a stationary process with spectral density

$$s_U(\omega) = \left[|G|^2 s_Z\right](\omega). \tag{6}$$

If $G(\omega)$ is nonzero on Δ, Δ is the spectral support of $\mathbf{Z}$ and $\mathbf{U}$ in the same time. In this circumstance, the FSE (Fourier Series Expansion) of $e^{i\omega t}$ on Δ [9]

$$e^{i\omega t} = \sum_{n\in\mathbb{Z}} \alpha_n(t) e^{i\omega t_n}, \omega \in \Delta \tag{7}$$

leads at the same time to the following relationships:

$$\left\{ \begin{array}{l} U(t) = \sum_{n\in\mathbb{Z}} \alpha_n(t) U(t_n) \\ Z(t) = \sum_{n\in\mathbb{Z}} \alpha_n(t) Z(t_n). \end{array} \right. \tag{8}$$

Roughly speaking, an interpolation formula is equivalent to a generalized Fourier series of $e^{i\omega t}$ convergent on the spectral support. In practice, only a finite number of samples are available and the sums in (8) are truncated. If $G(\omega)$ is a nonzero function on interval Δ, we can define a LIF $\mathcal{G}^{-1}$ such that

$$\left\{ \begin{array}{l} Z(t) = \mathcal{G}^{-1}[\mathbf{U}](t) = \int_{-\infty}^{\infty} g_{-1}(t-u) U(u) du \\ g_{-1}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{G}_{-1}(t) e^{-i\omega t} dt \\ \overline{G}^{-1}(\omega) = \left\{ \begin{array}{c} 1/G(\omega), \omega \in \Delta \\ 0, \omega \notin \Delta. \end{array} \right. \end{array} \right. \tag{9}$$

$\overline{G}^{-1}(\omega)$ and $g_{-1}(t)$ are the complex gain and the impulse response of $\mathcal{G}^{-1}$. We have also

$$Z(t) = \mathcal{G}^{-1}[\mathbf{U}](t) = \int_{-\infty}^{\infty} g_{-1}(t-u) \left[\sum_{n\in\mathbb{Z}} \alpha_n(u) U(t_n)\right] du.$$

Changing the order of summations, we obtain the sampling formula

$$Z(t) = \sum_{n\in\mathbb{Z}} \left[\int_{-\infty}^{\infty} g_{-1}(t-u) \alpha_n(u) du\right] U(t_n) \tag{10}$$

The error consisting in using samples $U(t_n)$ as estimations of $Z(t_n)$ can be expressed as

$$\sigma^2 = \mathrm{E}\left[|Z(t) - U(t)|^2\right] = \int_{\Delta} |1 - G(\omega)|^2 s_Z(\omega) d\omega. \tag{11}$$

C. Interpolation with non uniform samples using PNS2L

In the particular case of periodic sampling of a baseband signal with $t_n = n/f_s$, coefficients $\alpha_n(t)$ can be deduced from the well known Shannon reconstruction formula:

$$\alpha_n(t) = \frac{\sin(\pi f_s(t - t_n))}{\pi f_s(t - t_n)} \tag{12}$$

In this section, we consider the more general case of a spectral support not necessarily centered on 0. Moreover, we assume an irregular sampling of the process $\mathbf{U} = \{U(t), t \in \mathbb{R}\}$ (non periodic but known time instants) and we show how to derive coefficients $\alpha_n(t)$ in this case, using previously published PNS$2L$ [7].

Let define frequency bands $\Delta_k, 1 \le |k| \le L$ as

$$\Delta_k = \left\{ \begin{array}{l} \omega_0 + \left[\frac{(k-1)\pi}{L}, \frac{k\pi}{L}\right[, k \ge 1 \\ \omega_0 + \left]\frac{k\pi}{L}, \frac{(k+1)\pi}{L}\right], k \le -1 \end{array} \right. . \tag{13}$$

Intervals Δ_k define a partition of interval Δ (ω_0 is the pulsation corresponding to the center of spectral band Δ) in $2L$ intervals of length $\frac{\pi}{L}$. Defining $\mathbf{U}_k = \{U_k(t), t \in \mathbb{R}\}$ as the result of an ideal bandpass filtering on Δ_k, the following equality holds:

$$U(t) = \sum_{1\le|k|\le L} U_k(t). \tag{14}$$

Using this decomposition, equation (15) is demonstrated in [7] based on PNS$2L$ sampling plan (Periodic Nonuniform Sampling of order $2L$ [10], [11]).

$$\begin{array}{c} \sum_{1\le|k|\le L} U_k(t) e^{-i[\gamma_k(t-\theta)+\omega_0 t]} = \sum_{j\in\mathbb{Z}} (-1)^j \ldots \\ \ldots \mathrm{sinc}\left[\frac{\pi}{2L}(t-\theta) - \pi j\right] U(\theta + 2jL) e^{-i\omega_0(\theta+2jL)} \\ \gamma_k = \frac{\pi}{L}\left(k - \frac{1}{2}\mathrm{sgn}(k)\right). \end{array} \tag{15}$$

for all real number θ and where $\mathrm{sgn}(k) = k/|k|$, $\mathrm{sinc}(x) = \sin(x)/x$.

The measured quantities $U(t_n)$ appear in the right-hand side of (15) for $j = 0, \theta = t_n$. If L is large enough and provided that t is in the neighborhood of the observation interval $[t_{-L}, t_L]$, it is possible to neglect the terms for $j \neq 0$ in the right-hand side of (15) (sinc function converges to 0):

$$\begin{array}{c} \sum\limits_{1\leq|k|\leq L} \widehat{U}_k\left(t\right) e^{-i\left[\gamma_k\left(t-t_n\right)+\omega_0 t\right]} = ... \\ ...\text{sinc}\left[\frac{\pi}{2L}\left(t-t_n\right)\right] U\left(t_n\right) e^{-i\omega_0 t_n} \end{array} \tag{16}$$

where $\widehat{U}_k(t)$ is an estimation of $U_k(t)$. Then, for a given value of t, varying the parameter n over the interval $-L, ..., -1, 1, ..., L$, a linear system with $2L$ equations and $2L$ unknowns can be obtained. Rather than considering that the unknowns are the terms $(\widehat{U}_k\left(t\right), 1 \leq |k| \leq L)$ in (16), it is easier to consider that the unknowns are the terms $\left\{\widehat{U}_k\left(t\right) e^{-i\left(\gamma_k+\omega_0\right)t}, 1 \leq |k| \leq L\right\}$ so that the matrix M to be inverted does not depend on time t (only one inversion has to be performed as time is varying):

$$M\underline{x}(t) = \underline{b}(t) \tag{17}$$

with

$$M = \left[\begin{array}{ccc} e^{i\gamma_{-L}t_{-L}} & ... & e^{i\gamma_L t_{-L}} \\ \vdots & \ddots & \vdots \\ e^{i\gamma_{-L}t_L} & ... & e^{i\gamma_L t_L} \end{array}\right],$$

$$\underline{x}(t) = \left[\begin{array}{l} \widehat{U}_{-L}(t)e^{-i(\gamma_{-L}+\omega_0)t} \\ \vdots \\ \widehat{U}_L(t)e^{-i(\gamma_L++\omega_0)t} \end{array}\right]$$

and

$$\underline{b}(t) = \left[\begin{array}{l} \text{sinc}\left[\frac{\pi}{2L}\left(t-t_{-L}\right)\right] U\left(t_{-L}\right) e^{-i\omega_0 t_{-L}} \\ \vdots \\ \text{sinc}\left[\frac{\pi}{2L}\left(t-t_L\right)\right] U\left(t_L\right) e^{-i\omega_0 t_L} \end{array}\right]$$

it is worth noting that matrix M depends only on non uniform time instants t_n. Once matrix M^{-1} is known (this inversion is required only one time because the matrix is independent of t), we can express each $\widehat{U}_k(t)$ as a linear combination of observations $U(t_n), n \in \{-L, ..., -1\} \cup \{1, ..., L\}$ using (17), allowing to express also $U(t)$ as a linear combination of observations $U(t_n)$ using (14), allowing to compute coefficients $\alpha_n(t)$ in (8).

III. Simulation results

The signal considered for simulations is a Binary Phase-Shift Keying (BPSK) telecommunication signal whose baseband representation $Z_b(t)$ is built by filtering a pulse train (Diracs separated by a symbol period T with random equiprobable sign ± 1) by a RCF (Raised Cosine Filter) with cut-off frequency $R = \frac{1}{T}$ (corresponding to the symbol rate) and roll-of factor $\beta = 0.25$ (18).

$$H(\omega) = \left\{\begin{array}{ll} T, & |\omega| \leq \frac{\pi(1-\beta)}{T} \\ \frac{T}{2}\left[1+\cos\left(\frac{T}{2\beta}\left[|\omega|-\frac{\pi(1-\beta)}{T}\right]\right)\right], & \frac{\pi(1-\beta)}{T} < |\omega| \leq \frac{\pi(1+\beta)}{T} \\ 0, & \text{otherwise} \end{array}\right. \tag{18}$$

Here, we assume that signal $Z(t) = Z_b(t)e^{i\omega_0 t}$ is available in IF (Intermediate Frequency): interval Δ is centered around a carrier frequency corresponding to pulsation $\omega_0 = 1.6 \times 2\pi R$. We choose empirically $R = 1$ Hz and we sample this signal at the minimum possible average sampling frequency $f_s = (1+\beta)R$ (total band of the signal), retaining $2L = 120$ samples. We assume that samples of $Z(t)$ are collected after an imperfect (low-cost) sampling device introducing voluntarily very high jitter (100%): we observe $Z(t)$ at time instants

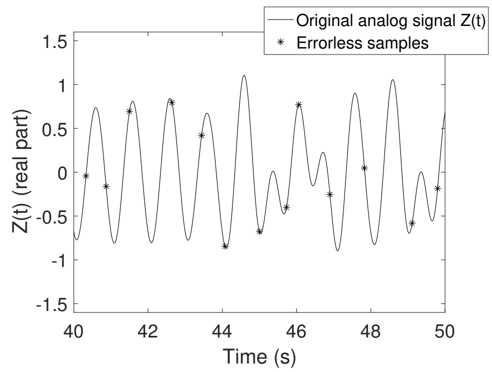

Fig. 1. Real part of the analyzed signal in IF.

$$t_n = nT_s + \theta, n \in \{-L, ..., -1\} \cup \{1, ..., L\} \tag{19}$$

where $T_s = \frac{1}{f_s}$ is the average sampling period and θ is uniformly distributed between $-\frac{T_s}{2}$ and $\frac{T_s}{2}$. The analog signal analyzed in these simulations is depicted in figure 1 and considered nonuniform sampling instants are marked with stars (only the real part of the signal is plotted).

The impulse response of the sampling device is also imperfect. In this example, we address two different sampling shapes (with unit area)

$$\left\{\begin{array}{ll} \text{R: Rectangular} & g\left(t\right) = \frac{1}{2\varepsilon}, |t| < \varepsilon \\ \text{G: Gaussian} & g\left(t\right) = \frac{1}{\varepsilon\sqrt{2\pi}} \exp\left[-t^2/2\varepsilon^2\right] \end{array}\right. \tag{20}$$

corresponding to complex gains

$$\left\{\begin{array}{ll} \text{R: Rectangular} & G\left(\omega\right) = \text{sinc}\left(\varepsilon\omega\right) \\ \text{G: Gaussian} & G\left(\omega\right) = \exp\left(-\varepsilon^2\omega^2\right) \end{array}\right. \tag{21}$$

where $\text{sinc}x = \frac{\sin x}{x}$. Those two cases are examples of approximation models of real sampling devices. In the first case, the value of ε has to be such that $G\left(\omega\right)$ has no zero into Δ, the spectral support of $Z(t)$.

For the cases Rectangular (R) and Gaussian (G), (11) yields for a white noise with $s_Z\left(\omega\right) = \frac{1}{2\pi}, \omega \in [-\pi, \pi]$

$$\begin{array}{l} \sigma_R^2 = \frac{1}{2\pi} \int\limits_{\omega_0-\pi}^{\omega_0+\pi} \left(1-\text{sinc}\left(\varepsilon\omega\right)\right)^2 d\omega \\ \sigma_G^2 = \frac{1}{2\pi} \int\limits_{\omega_0-\pi}^{\omega_0+\pi} \left(1-\exp\left(-\varepsilon^2\omega^2\right)\right)^2 d\omega \end{array} \tag{22}$$

IF sampling receivers (violating Nyquist criteria) can be found on the market [12]. For this kind of devices, Signal to Noise Ratio (SNR) is known to decrease when the frequency of the input analog signal increases. It is likely to be due to the lowpass behavior of the sampler: as can be seen in (22) in the particular case of a white noise: σ_R^2 and σ_G^2 increase as ω_0 increases.

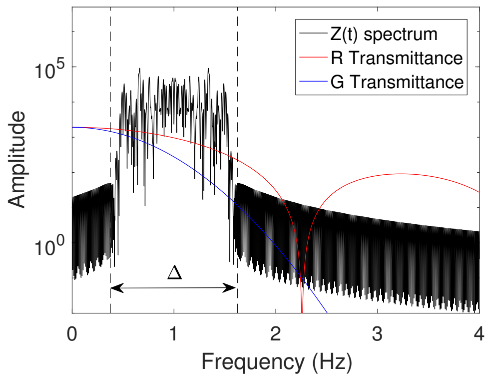

Fig. 2. Signal spectrum and transmittances of the samplers.

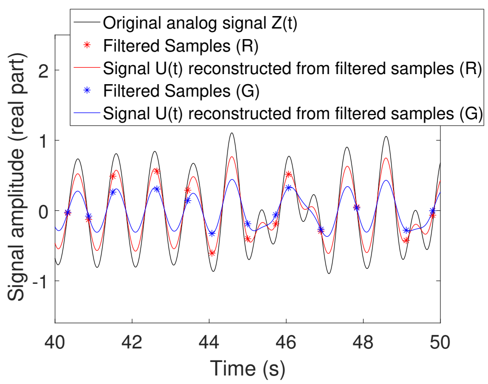

Fig. 3. Real part of signal reconstructed with filtered samples using PNS2L classical formula (16).

In this example, $\varepsilon = 0.221$ s for both rectangular (R) sampler and Gaussian (G) sampler. The spectrum of $Z(t)$ and the 2 transmittances (R and G) representing the assumed shapes of the sampler are depicted figure 2.

Because of this non ideal sampling shape, reconstructed signal is attenuated (and distorted) if we try to reconstruct it directly using (16) from filtered samples coming from the signal in IF (figure 3). Note that the distortion would have been much lower if we had tried to sample the baseband signal rather then the IF one.

But if we make use of corrected formulas taking into account the shape of the sampler using (10), the reconstruction of the IF signal is almost perfect as can be seen figure 4 (all curves are superimposed). It is then very simple to retrieve undistorted BPSK symbols (at 1 sample by symbol) if needed.

IV. Conclusion

To summarize, we assumed that the sampling process is defined by two main components: a sequence $\mathbf{t} = \{t_n, n \in \mathbb{Z}\}$ of real numbers, and a regular function $g(t)$ such that the convolution product $[g * Z](t_n)$ defines observed data (instead of perfect sampling $Z(t_n) = [\delta * Z](t_n)$ where δ is the Dirac distribution). More precisely, we observe distorted samples $U(t_n)$ where $\mathbf{U} = \mathcal{G}[\mathbf{Z}]$ is the output of the LIF with input $\mathbf{Z}$, impulse response $g(t)$ and complex gain $G(\omega)$. The inverse

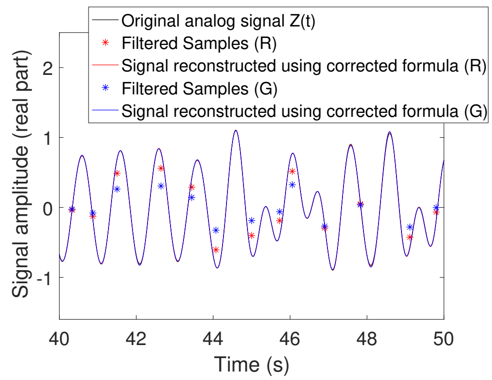

Fig. 4. Real part of signal reconstructed with filtered samples using PNS2L corrected formula (10).

filter $\mathcal{G}^{-1}$ is defined by its complex gain $G^{-1}(\omega)$ (on the frequency interval Δ corresponding to the spectral support of process $\mathbf{Z}$) and its impulse response $g_{-1}(t)$. As soon as we have a sampling formula for $\mathbf{U}$ fitted to the data $U(t_n)$ (as for instance (16)), we have also a sampling formula for $\mathbf{Z}$ fitted to the data $U(t_n)$, defined by (10). Finally, we also derived the expression of the error obtained by taking $U(t)$ as estimation of $Z(t)$ (equation (11)).

An example of a telecommunication signal in IF is studied with an arbitrary underlying sampling sequence $\mathbf{t}$ (actually periodic with any jitter), with two examples of shape functions $g(t)$ (Rectangular or Gaussian). Results demonstrate the ability to save hardware complexity needed to down-convert the IF signal into baseband (2 mixers and one filter) by adding moderate software complexity (only one matrix inversion is needed).

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