NEW INDICES OF COHERENCE FOR ONE AND TWO-DIMENSIONAL FIELDS

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ABSTRACT

The modern definition of optical coherence highlights a frequency dependent function based on a matrix of spectra and cross-spectra. Due to general properties of matrices, such a function is invariant in changes of basis. In this article, we attempt to measure the proximity of two stationary fields by a real and positive number between 0 and 1. The extremal values will correspond to uncorrelation and linear dependence, similar to a correlation coefficient which measures linear links between two random variables. We show that these "indices of coherence" are generally not symmetric, and not unique. We study and we illustrate this problem together for one-dimensional and two-dimensional fields in the framework of stationary processes.

keywords: Coherence, optical beams, stationary processes, linear invariant filters.

1. INTRODUCTION

Originally, the coherence of an optical beam measured its ability to interfere. The beam can be modelled by a "quasi-monochromatic" one-dimensional process, which means that the power spectrum is close to a line at some frequency $\omega_0/2\pi$. We know that a thin spectral line allows a larger number of franges than a line with a larger width. The spectral width is related to the auto-correlation function which decreases faster when the width increases. In this simple situation, the coherence $\gamma(\tau)$ can be defined as the reduced auto-correlation function, and then takes the value 1 for $\tau = 0$, and the value 0 for large values of τ, except for the ideal monochromatic wave, which is always an idealization.

When comparing two one-dimensional processes (the model is now two-dimensional), the "complex degree of coherence" $\gamma_{12}(\tau)$ is defined by [1], [2]

$$\gamma_{12}(\tau) = \frac{K_{12}(\tau)}{\sqrt{K_1(0)\, K_2(0)}} \tag{1}$$

where K_1, K_2, K_{12} are the auto-correlations and the cross-correlation between both processes. We have $0 \leq |\gamma_{12}(\tau)| \leq 1$ by the Cauchy-Schwarz inequality, but it is possible that $\gamma_{12}(\tau)$ does not reach the value 1, and the maximal value has a particular sense. Optical beams have a two-dimensional electrical field orthogonal to the trajectory. The "mutual coherence" between two points P_1 and P_2 is defined from the "electric mutual coherence matrix" of correlation and cross-correlation functions [3]

$$\mathbf{M}(\tau) = \begin{bmatrix} K_{11}(\tau) & K_{12}(\tau) \\ K_{21}(\tau) & K_{22}(\tau) \end{bmatrix} \tag{2}$$

where $K_{ij}(\tau) = \mathrm{E}\left[E_i(P_1, t)\, E_j^*(P_2, t-\tau)\right]$, $E_i(P_k, t)$ being the component i $(i = 1, 2)$ of the field at the point P_k $(k = 1, 2)$ in some orthogonal basis. The "complex degree of coherence" is defined by (I_1 and I_2 are the intensities) [2]

$$\gamma(\tau) = \frac{\mathrm{tr}\mathbf{M}(\tau)}{\sqrt{I_1 I_2}}. \tag{3}$$

It is worth noting that this quantity does not depend on the orthogonal basis of reference because $\mathrm{tr}\mathbf{M}(\tau), I_1, I_2$ are matricial invariants [4].

Rather than working with correlations, modern optics consider spectral and cross-spectral matrices [3]. The "spectral degree of coherence" has the same shape as (3) except that it is a function of the frequency $\omega/2\pi$ through the "electric cross-spectral density". Whatever the framework, the "degree" of coherence is a real or complex quantity which may depend on the time or on the frequency but which may be constant. It is the case in a number of domains of physical or human sciences from astrophysics to demography. Here, its modulus takes the extreme values 0 and 1 in very particular circumstances of linkage between the processes taken into account.

Formulas (1) and (3) were fitted to interferences plans. For values τ_0 such that $\gamma_{12}(\tau_0) = 1$, powers of processes are added for the delay τ_0. Moreover, this means that both processes taken into account can be deduced by a linear operation characterized by a Linear Invariant Filter (LIF) (see the following section). Conversely, if $|\gamma_{12}(\tau)|$ cannot take the value 1, this means that some parts of both processes are uncorrelated, even if other parts are very strongly linked. It is clear that the latter parts can lead to interferences from matched devices, but not the first parts. Clearly, it is very important to study this kind of decomposition, and to deduce measures of "distances" between processes. We will deduce a reasonable family of "indices of coherence".

Let assume that we look at two beams defined by three uncorrelated processes **A, B** and **C.** The beams

$$\mathbf{D} = \mathbf{A} + \mathbf{B} \text{ and } \mathbf{E} = \mathbf{A} + \mathbf{C} \tag{4}$$

can generate good plans of interference only when the "intensities" of **B** and **C** are weak with respect to **A.** In such a decomposition, we see the parts which interfere (**A** with itself) and the parts which cannot interfere (**B** with **C** and **A** for instance). Conversely, to provide such a decomposition gives strong informations on the ability of beams to interfere.

The notion of coherence has to be put in front of neighboorhood, proximity, distance, common points between functions, random variables, random processes, or family of random processes. In the framework developed here, Hilbert spaces of random variables and linear algebra allow the simplest theoretical developments.

In this article, we look for an "index of coherence" which expresses the proximity of some processes. It will be a positive number between 0 and 1, the extremal values being reserved to the uncorrelation and the total dependence. The next section studies the one-dimensional case, where we look for links between two one-dimensional processes. The third section provides a generalization to two-dimensional random processes. Simple examples are developed in both cases. Appendices recall used mathematical tools and too long proofs.

2. ONE-DIMENSIONAL CASE

2.1. A family of indices of coherence

1) Let **U, V** be two stationary random processes with elements $U(t), V(t), t \in \mathbb{R}$, power spectral densities $s_U(\omega), s_V(\omega)$, cross-spectrum $s_{UV}(\omega)$ (see appendix 1). We assume that the supports of $s_U(\omega)$ and $s_V(\omega)$ are identical.

Let consider the linear invariant filter (LIF) $\mathcal{F}$ with complex gain $\phi(\omega)$ defined by (see appendix 1)

$$\phi(\omega) = \left[\frac{s_{VU}}{s_U}\right](\omega). \tag{5}$$

When the processes **A** and **B** verify

$$\left\{ \begin{array}{c} A(t) = \mathcal{F}[\mathbf{U}](t) \\ V(t) = A(t) + B(t) \end{array} \right. \tag{6}$$

the processes **A** and **B** become orthogonal (E$[A(t)B^*(u)] = 0$) and

$$\left\{ \begin{array}{c} A(t) \in \mathbf{H}_U, B(t) \perp \mathbf{H}_U \\ \mathrm{E}\left[|V(t)|^2\right] = \mathrm{E}\left[|A(t)|^2\right] + \mathrm{E}\left[|B(t)|^2\right] \end{array} \right. \tag{7}$$

where $\mathbf{H}_U$ is the Hilbert space of linear combinations of the $U(t)$ when t spans $\mathbb{R}$ (see appendix 2).

Formula (6) splits $V(t)$ into two parts, the first one $A(t)$ which belongs to $\mathbf{H}_U$ (it is a linear combination of the $U(u), u \in \mathbb{R}$), and the second one $B(t)$ which is orthogonal to $\mathbf{H}_U$ (i.e. uncorrelated with the $U(u), u \in \mathbb{R}$).

In decomposition (6), **V** and **U** are "neighbouring" when $E\left[|B(t)|^2\right]$ is weak compared to $E\left[|A(t)|^2\right]$, which is equivalent to a strong "coherence" between them. Conversely, the "coherence" is weak when this ratio is too large. In this context, the "distance" between **V** and **U** is not defined by a accurate relation taking into account the r.v. $U(t)$ and $V(t)$, but a "distance" between for instance $U(t)$ and $\mathbf{H}_V$ (spanned by the set $V(u), u \in \mathbb{R}$). These considerations allow to define "indices of coherence" which are constants and not some functions of the time or the frequency. Let consider ρ_{UV} defined by

$$\begin{array}{c} \rho_{UV} = \frac{\mathrm{E}\left[|A(t)|^2\right]}{\mathrm{E}\left[|V(t)|^2\right]} = \frac{1}{\sigma_V^2}\int_{-\infty}^{\infty}\left[\frac{|s_{VU}|^2}{s_U}\right](\omega)\,d\omega \\ \sigma_V^2 = K_V(0) = \int_{-\infty}^{\infty} s_V(\omega)\,d\omega. \end{array} \tag{8}$$

We have $\rho_{UV} \in [0,1]$, and

$$\left\{ \begin{array}{c} \rho_{UV} = 1 \Longleftrightarrow V(t) = \mathcal{F}[\mathbf{U}](t) \\ \rho_{UV} = 0 \Longleftrightarrow V(t) \perp \mathbf{H}_U. \end{array} \right. \tag{9}$$

In the first case, the process **V** can be (linearly) reconstructed from **U,** and both processes **U** and **V** are orthogonal in the second case. $A(t)$ is the part of $V(t)$ which is explained by **U** i.e. by the set of random variables $U(u), u \in \mathbb{R}$ together than ρ_{UV} is a measure of the part of the power of $V(t)$ which is explained by **U**.

2) In the case $\rho_{UV} = 1$ ($\mathbf{B} = \mathbf{0}$), both processes are "coherent" (**U** defines completely **V** and conversely), and "uncoherent" when $\rho_{UV} = 0$ (**U** and **V** are orthogonal). For $\rho_{UV} \neq 0$ and 1, they are "partially coherent". It is useless to add the redundant terms "totally" or "completely" or other qualifiers [5], [6]. Clearly, ρ_{UV} has the qualities that we expect for an "index of coherence". The fact that ρ_{UV} is deduced from a perfectly defined orthogonal decomposition is a strong supplementary argument. It is worth noting that, using $(6), (7)$

$$\rho_{\mathcal{G}[\mathbf{U}]V} = \rho_{UV} \tag{10}$$

where $\mathcal{G}$ is some LIF (with complex gain which does not cancel). This equality shows that ρ_{UV} measures the proximity of $\mathbf{H}_U$ with $V(t)$. We obtain the same value of the index of coherence whatever the stationary process chosen in $\mathbf{H}_U$.

Unfortunately, this index is not symmetric, except for the bounds 0 and 1. We have

$$\left\{ \begin{array}{c} \rho_{UV} - \rho_{VU} = \int_{-\infty}^{\infty}\left[\frac{|s_{VU}|2}{s_U s_V}\upsilon\right](\omega)\,d\omega \\ \upsilon(\omega) = \left[\frac{s_V}{\sigma_V^2} - \frac{s_U}{\sigma_U^2}\right](\omega) \end{array} \right.$$

This quantity has no reason to cancel, except if we add hypotheses, for instance the equality of the normalized spectra. Moreover, the family of ρ_a defined by

$$\rho_a = a\rho_{UV} + (1-a)\,\rho_{VU} \tag{11}$$

verifies the conditions above when $a \in [0,1]$. Then, it is easy to construct families of positive numbers which illustrate links between two stationary processes. Obviously, $\rho_{1/2}$ is symmetric, and it is the only one symmetric provided that $\rho_{UV} \neq \rho_{VU}$.

3) Now, let assume that the support of s_U (and s_V) can be split in the sets Δ and Δ' of positive measure such that

$$\begin{cases} s_{UV}(\omega) = 0, \omega \in \Delta \\ s_{UV}(\omega) \neq 0, \omega \in \Delta'. \end{cases}$$

In decomposition (6) we have at the same time

$$\begin{cases} s_A(\omega) = \left[\frac{|s_{VU}|^2}{s_U}\right](\omega) = 0, \omega \in \Delta \\ s_A(\omega) > 0, \omega \in \Delta' \end{cases}$$

which implies $0 < \rho_{UV} < 1$. By symmetry, it is the same for ρ_{VU}. Obviously, we find the same result when the supports of s_U and s_V are distinct.

4) Let assume that **U** and **V** model an optical beam, where a source, a direction and a sense of propagation are given. If **U** is nearer the source than **V,** the decomposition (6) is natural, because we can consider that **U** is a source for **V**. In the latter, we expect to find a component closely linked to **U** added to a noise which models a loss of information. **A** and **B** answer this question. This point of view is accurately expressed by (7) and by the index ρ_{UV} of (8) rather than by ρ_{VU} based on a decomposition of $U(t)$.

Whatever the definition of the index of coherence, based on the decomposition of one or both processes **U** and **V**, it is clear that the decomposition itself in two processes (for instance **A** and **B**) gives more insights about links between the processes than any index only based on statistical moments. When the beam is not reduced to a ray but fills some volume close to some axis, the relative places of the processes are more difficult to characterize.

To summarize, we have defined a real and positive index ρ_{UV} which takes its values in $[0,1]$ and which expresses the proximity between **U** and **V.** We will say that both processes are "coherent" when $\rho_{UV} = 1$, and "uncoherent" when $\rho_{UV} = 0$ (rather than fully or completely coherent or uncoherent). In other cases, they will be "partially coherent". Actually, through (11), we have shown that ρ_{UV} and ρ_{VU} define a family of indices ρ_a $(0 \leq a \leq 1)$ with the same properties, added to the symmetry property for $\rho_{1/2}$.

2.2. Estimation and index of coherence

When the LIF $\mathcal{F}^{-1}$ is well defined, (6) is equivalent to

$$\begin{cases} W(t) = \mathcal{F}^{-1}[\mathbf{V}](t) = U(t) + C(t) \\ C(t) = \mathcal{F}^{-1}[\mathbf{B}](t) \\ U(t) = \mathcal{F}^{-1}[\mathbf{A}](t). \end{cases}$$

Both processes **U** and **C** are uncorrelated. We look for an estimation of **U** from the observation of **V** (equivalently from the observation of **W**)**.** The "Wiener filter" $\mathcal{N}$ with input **W,** output $\widetilde{\mathbf{U}}$ and complex gain $\eta(\omega)$, is classically defined by

$$\eta = \left[\frac{s_U}{s_W}\right] = \frac{|s_{VU}|^2}{s_U s_V}$$

$\widetilde{U}(t)$ is an estimator of $U(t)$ based on the observation of **V,** with the mean-square error

$$\mathrm{E}\left[\left|U(t) - \widetilde{U}(t)\right|^2\right] = \int_{-\infty}^{\infty}\left[s_U\left(1 - \frac{|s_{VU}|^2}{s_U s_V}\right)\right](\omega)\,d\omega.$$

The normalized error is

$$\frac{1}{\sigma_U^2}\mathrm{E}\left[\left|U(t) - \widetilde{U}(t)\right|^2\right] = 1 - \rho_{VU}.$$

This last equality links the index of coherence ρ_{VU} with the relative error in the mean-square estimation of the process **U** from the observation of the process **V**. The errorless reconstruction corresponds to $\rho_{VU} = 1$, and the worse one to $\rho_{VU} = 0$, as expected.

2.3. Another symmetric index of coherence

The "spectral degree of coherence at the frequency f" of a scalar field is currently defined by [2], [7] p. 170,

$$\mu_{UV}^0(\omega) = \left[\frac{|s_{UV}|^2}{s_U s_V}\right](\omega)$$

which depends on the frequency $f = \omega/2\pi$. The quantity which is sought herein is a constant. The usual method for reaching this aim is an integration on the frequency axis. But nothing can assert that

$$\mu_{UV}^1 = \int_{-\infty}^{\infty} \mu_{UV}^0(\omega)\,d\omega$$

verifies the conditions verified by ρ_{UV}. As an example, $\mu_{UV}^1 = \infty$ in example 1 of section 2.4.1 with

$$s_U(\omega) = e^{-|\omega|}, s_N(\omega) = e^{-2|\omega|}.$$

For an index which has to characterize a global behavior, the places which hold a larger power have to be emphazised.

For instance, if we weight the integral which defines μ_{UV}^1 by s_V/σ_V^2, we obtain the finite index defined in (8)

$$\rho_{UV}=\frac{1}{\sigma_V^2}\int_{-\infty}^{\infty}\left[\mu_{UV}^0 s_V\right](\omega)\,d\omega$$

but this index is not symmetric ($\rho_{UV}\neq\rho_{VU}$ most of the time). The symmetry condition is verified by

$$\int_{-\infty}^{\infty}\left[\frac{|s_{UV}|^2}{\sqrt{s_U s_V}}\right](\omega)\,d\omega$$

which is a finite quantity because

$$\int_{-\infty}^{\infty}\left[\sqrt{s_U s_V}\right](\omega)\,d\omega<\infty$$

from the Schwarz inequality. A normalization leads to a new symmetric index of coherence

$$\mu_{UV}=\int_{-\infty}^{\infty}\left[\frac{|s_{UV}|^2}{\sqrt{s_U s_V}}\right](\omega)\,d\omega\Big/\int_{-\infty}^{\infty}\left[\sqrt{s_U s_V}\right](\omega)\,d\omega \tag{12}$$

because μ_{UV} is a real and positive quantity such that $0\leq\mu_{UV}\leq 1$, $\mu_{UV}=0$ if and only if $s_{UV}=0$ ($\mathbf{U}$ and $\mathbf{V}$ are uncoherent), and $\mu_{UV}=1$ if and only if **U** and **V** are coherent. When μ_{UV} is different from 0 and 1, it is also true for ρ_{UV} and ρ_{VU}. The values of these last quantities are linked to some information held by the process **U** about $V(t)$ (or the converse) through mean-square estimations. We do not have such a meaning for μ_{UV}, which is only built to fulfill some mathematical conditions of normalization and symmetry.

2.4. Examples

2.4.1. Example 1

The simplest model of transmission verifies

$$\mathbf{V}=\mathbf{U}+\mathbf{N}$$

where **N** is a "noise" uncorrelated with the "signal" **U**. About the decomposition (6), we find (in accordance with intuition)

$$\mathbf{U}=\mathbf{A},\mathbf{N}=\mathbf{B}$$

which leads to the index of coherence

$$\rho_{UV}=\frac{\sigma_U^2}{\sigma_U^2+\sigma_N^2}.$$

The converse is different. ρ_{VU} is obtained from the equations (we rewrite $(5),(6),(8)$ inverting **U** and **V**)

$$\left\{\begin{array}{c}\phi=\frac{s_{UV}}{s_V}=\frac{s_U}{s_U+s_N}\\ A(t)=\mathcal{F}[\mathbf{V}](t)\\ U(t)=A(t)+B(t).\end{array}\right.$$

which leads to (using (8))

$$\rho_{VU}=\frac{1}{\sigma_U^2}\int_{-\infty}^{\infty}\left[\frac{s_U^2}{s_U+s_N}\right](\omega)\,d\omega.$$

In both situations, we find 1 and 0 as limits following the "ratio" between the "signal" and the "noise". Between these limits, values of ρ_{UV} and ρ_{VU} are generally different. As an example, let assume that ($a>0$)

$$\left\{\begin{array}{c}s_N(\omega)=1-|\omega|,\omega\in(-1,1)\text{ and 0 elsewhere}\\ s_U(\omega)=a,\omega\in(-1,1)\text{ and 0 elsewhere}\end{array}\right.$$

which yields

$$\left\{\begin{array}{c}\rho_{UV}=\frac{2a}{1+2a},\rho_{VU}=a\ln\left(1+\frac{1}{a}\right)\\ \mu_{UV}=3a\frac{\sqrt{1+a}-\sqrt{a}}{(1+a)^{3/2}-a^{3/2}}.\end{array}\right.$$

Figure 1 compares the indices as a function of a. The three curves verify the limit conditions (0 for $a=0$ and 1 for $a=\infty$), and are not very different for other values of a.

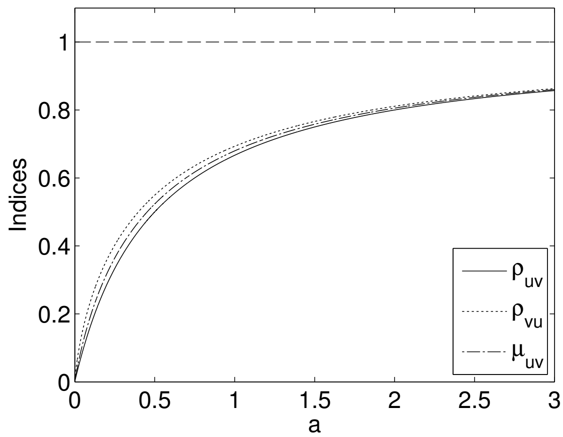

Fig. 1. Example 1 (section 2.4.1), ρ_{uv}, ρ_{vu}, μ_{uv} versus a.

2.4.2. Example 2

Let **X** be a real process independent of **U** with characteristic functions (in the probability calculus sense)

$$\left\{\begin{array}{c}\alpha(\omega)=\mathrm{E}\left[e^{-i\omega X(t)}\right]\\ \beta(\tau,\omega)=\mathrm{E}\left[e^{-i\omega(X(t)-X(t-\tau)}\right]\end{array}\right. \tag{13}$$

independent of t, which implies that **X** is stationary in a sense stronger than the usual second order one. We define **V** by

$$V(t)=U(t-X(t)).$$

$X(t)$ models a random propagation time which can take into account variations of the refraction index or other hazards [8], [9]. Easy computations lead to

$$\left\{\begin{array}{c}s_{UV}=\alpha^* s_U\Longrightarrow\phi=\alpha\\ \rho_{UV}=\frac{1}{\sigma_U^2}\int_{-\infty}^{\infty}\left[|\alpha|^2 s_U\right](\omega)\,d\omega.\end{array}\right. \tag{14}$$

Moreover,

$$K_V(\tau) = \int_{-\infty}^{\infty} e^{i\omega\tau} \beta(\tau, \omega) s_U(\omega) d\omega$$

which allows the determination of $s_V(\omega)$ from a Fourier transform. Let assume that

$$s_U(\omega) = \frac{1}{2\pi}, \omega \in (-\pi, \pi) \text{ and 0 elsewhere}$$

and that **X** is a telegraph signal with values $\pm a$ and parameter λ (which rules the rate of polarity changes) [10]. In this situation

$$\begin{cases} \alpha(\omega) = \cos a\omega \\ \beta(\tau, \omega) = \cos^2 a\omega + e^{-2\lambda|\tau|} \sin^2 a\omega. \end{cases}$$

From (8), we obtain

$$\rho_{UV} = \frac{1}{2}\left(1 + \frac{\sin 2a\pi}{2a\pi}\right)$$

which does not depend on λ, and varies from 1 ($a = 0$) to 1/2 ($a = \infty$). Even for large a, $V(t + a)$ (or $V(t - a)$) provides an estimation of $U(t)$ which is errorless approximately half of the time. Moreover, using the convolution theorem

$$\begin{cases} s_V(\omega) = \frac{\cos^2 a\omega}{2\pi}\Omega(\omega) + \frac{1}{\pi^2}\int_{-\pi}^{\pi} \frac{\lambda \sin^2 au}{4\lambda^2 + (\omega - u)^2} du \\ \Omega(\omega) = 1, \omega \in (-\pi, \pi) \text{ and 0 elsewhere} \end{cases}$$

which depends on λ, and which never cancels. From (8)

$$\rho_{VU} = \int_{-\pi}^{\pi} \frac{\cos^2 a\omega}{4\pi^2 s_V(\omega)} d\omega.$$

Figure 2 shows variations of ρ_{VU} as function of λ (for $a = 0.1, 0.2, 0.3, 0.5, 1, 2, 4$). Figure 3 depicts variations of ρ_{UV} and ρ_{VU} in function of a for $\lambda = 4, 8, 16$. As explained above, ρ_{UV} and ρ_{VU} are not equal (except for the value 1).

2.4.3. *Example 3*

The lack of symmetry is obvious when spectral supports are not identical. Let assume that $s_V(\omega)$ does not cancel and that **U** is the the result of the low-pass filter with input **V,** and complex gain

$$\theta(\omega) = 1, \omega \in (-b, b) \text{ and 0 elsewhere.}$$

In this case, with respect to (6), we have $\mathbf{A} = \mathbf{U}$ and **B** is the output of a high-pass filter with input **V,** and complex gain

$$\theta'(\omega) = 0, \omega \in (-b, b) \text{ and 1 elsewhere.}$$

From (8) we deduce

$$\rho_{UV} = \frac{1}{\sigma_V^2}\int_{-b}^{b} s_V(\omega) d\omega$$

which verifies $\rho_{UV} < 1$, but we have $\rho_{VU} = 1$ because **U** is obtained from **V** through a LIF.

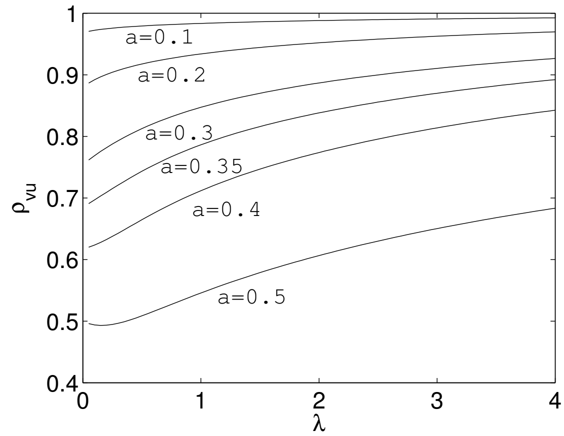

Fig. 2. Example 2 (section 2.4.2), ρ_{vu} for $a = 0.1, 0.2, 0.3, 0.35, 0.4$ and 0.5 versus λ.

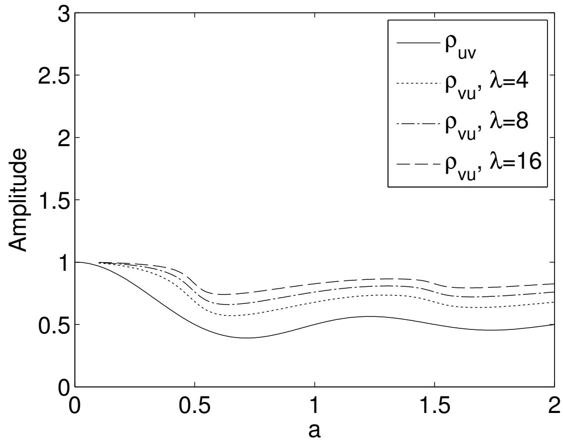

Fig. 3. Example 2 (section 2.4.2), ρ_{uv} and ρ_{vu} for $\lambda = 4, 8$ and 16.

2.4.4. *Example 4*

1) Let **U** be a normalized Gaussian process and **V** defined by

$$V(t) = \begin{cases} 1 \text{ when } U(t) > 0 \\ -1 \text{ when } U(t) < 0. \end{cases}$$

Results below are well-known [10]

$$\begin{cases} K_{UV}(\tau) = \sqrt{\frac{2}{\pi}} K_U(\tau) \\ K_V(\tau) = \frac{2}{\pi}\sin^{-1} K_U(\tau) \end{cases}$$

where $\sin^{-1}$ is the reciprocal function of the sine function. From (8) we deduce $\left(K_U(0) = \sigma_U^2 = 1\right)$

$$\rho_{UV} = \frac{2}{\pi}, \rho_{VU} = \frac{2}{\pi}\int_{-\infty}^{\infty}\left[\frac{s_U^2}{s_V}\right](\omega) d\omega.$$

Both quantities have no reason to be equal. Moreover, **U** defines **V,** which, in the linear context, is equivalent to **B=0** in

(6) and $\rho_{UV} = 1$. If this result is false, it is because the transformation (the sign function) is nonlinear. This function is not bijective (**V** gives few informations about **U**), but it is not the reason for the small value of ρ_{UV}.

2) Let **V** be defined by

$$V(t) = e^{U(t)} - \sqrt{e}$$

which is equivalent to

$$U(t) = \ln\left[V(t) + \sqrt{e}\right]$$

provided that $V(t) > -\sqrt{e}$. Both processes are coherent in the sense that **V** holds informations sufficient for an exact reconstruction of **U** (and conversely), but not "linear informations". Easy computations lead to

$$s_{UV}(\omega) = s_U(\omega)\sqrt{e}.$$

Consequently, from (8)

$$\rho_{UV} = \frac{1}{e-1} \cong 0.58$$

and not 1. This counter-example highlights the limits of linear tools. The property of "coherence" depends on the used mathematical tools. In this example, both processes are "partially coherent" in the linear framework, but "coherent" in a wider context.

2.4.5. Example 5

It can happen that $0 < \rho_{UV} = \rho_{VU} < 1$. It is the case when

$$\mathbf{U} = \mathbf{X} + \mathbf{N}, \mathbf{V} = \mathbf{X} + \mathbf{M}$$

where **X** and $(\mathbf{N}, \mathbf{M})$ are uncorrelated with $s_M = s_N$ (the processes are assumed real and different from **0**).

3. TWO-DIMENSIONAL CASE

We know that an optical beam is defined by a support, a direction of propagation and an electrical field orthogonal to this direction. In this section, we consider two-dimensional processes $\mathbf{U} = (\mathbf{U}_x, \mathbf{U}_y)$ and $\mathbf{V} = (\mathbf{V}_x, \mathbf{V}_y)$ where the components are taken with respect to an orthogonal system $Oxyz$ where Oz is the direction of propagation. The four-dimensional process $(\mathbf{U}_x, \mathbf{U}_y, \mathbf{V}_x, \mathbf{V}_y)$ is assumed globally stationary with spectral and cross-spectral densities $s_{U_x}, ..., s_{V_y}, s_{U_xU_y}, ...$ and identical spectral supports. We have to define an "index of coherence" between 0 and 1, which measures the proximity of the two-dimensional fields **U** and **V**, and which does not depend on the system of coordinates Oxy. As done previously, we will deduce from one particular index a natural family of available indices of coherence.

3.1. A definition of the index of coherence

As in section 2, we look for the part of **V**=$(\mathbf{V}_x, \mathbf{V}_y)$ which is explained by **U**=$(\mathbf{U}_x, \mathbf{U}_y)$. As proved in appendix 3, we have the following decomposition of **V**

$$\left\{\begin{array}{c} \mathbf{V}_x = \mathbf{V}'_x + \mathbf{V}''_x, \mathbf{V}_y = \mathbf{V}'_y + \mathbf{V}''_y \\ V'_x(t), V'_y(t) \in \mathbf{H}_{U_x} + \mathbf{H}_{U_y} \\ V''_x(t), V''_y(t) \perp \mathbf{H}_{U_x} + \mathbf{H}_{U_y} \end{array}\right. \tag{15}$$

where an element of the set $\mathbf{H}_{U_x} + \mathbf{H}_{U_y}$ is the addition of an element of $\mathbf{H}_{U_x}$ with an element of $\mathbf{H}_{U_y}$ (they are the Hilbert spaces spanned by both processes $\mathbf{U}_x, \mathbf{U}_y$). $V'_x(t)$ for instance is a linear combination of the $U_x(u)$ and the $U_y(v)$, and is, in the mean-square sense, the quantity that we can construct from these r.v. and which is the nearest to $V_x(t)$. The parts $\mathbf{V}'_x$ and $\mathbf{V}'_y$ hold informations about $\mathbf{U}_x$ and $\mathbf{U}_y$ and not $\mathbf{V}''_x$ and $\mathbf{V}''_y$.

Appendix 3 shows that the couple of processes $\mathbf{V}' = \left(\mathbf{V}'_x, \mathbf{V}'_y\right)$ can be retrieved from five LIF $\mathcal{A}_{xx}, \mathcal{A}_{xy}, \mathcal{M}, \mathcal{F}_{yx}, \mathcal{F}_{yy}$, as depicted in figure 5, with complex gains

$$\left\{\begin{array}{c} \alpha_{xx} = s_{V_xU_x}/s_{U_x}, \alpha_{xy} = s_{V_yU_x}/s_{U_x} \\ \mu = s_{U_yU_x}/s_{U_x} \\ \phi_{yx} = \frac{1}{\Delta}\left[s_{V_xU_y}s_{U_x} - s_{V_xU_x}s_{U_xU_y}\right] \\ \phi_{yy} = \frac{1}{\Delta}\left[s_{V_yU_y}s_{U_x} - s_{V_yU_x}s_{U_xU_y}\right] \\ \Delta = s_{U_x}s_{U_y} - \left|s_{U_xU_y}\right|^2 = s_{U_x}s_D \\ D(t) = U_y(t) - \mathcal{M}\left[U_x\right](t) \end{array}\right. \tag{16}$$

where the processes $\mathbf{U}_x$ and **D** are orthogonal by construction. We obtain the decompositions (we omit the variable t)

$$\left\{\begin{array}{c} V_x = V'_x + V''_x = \mathcal{A}_{xx}\left[\mathbf{U}_x\right] + \mathcal{F}_{yx}\left[\mathbf{D}\right] + V''_x \\ V_y = V'_y + V''_y = \mathcal{A}_{xy}\left[\mathbf{U}_x\right] + \mathcal{F}_{yy}\left[\mathbf{D}\right] + V''_y \end{array}\right. \tag{17}$$

where, in both lines, the three terms at right are uncorrelated because, by construction

$$\left\{\begin{array}{c} D(t) \in \mathbf{H}_{U_x} + \mathbf{H}_{U_y}, D(t) \perp \mathbf{H}_{U_x} \\ V''_x(t), V''_y(t) \perp \mathbf{H}_{U_x} + \mathbf{H}_{U_y}. \end{array}\right.$$

Consequently, $V'_x(t)$ and $V'_y(t)$ are the best (mean-square) estimations of $V_x(t)$ and $V_y(t)$ on $\mathbf{H}_{U_x} + \mathbf{H}_{U_y}$. A reasonable definition of the index of coherence ρ_{UV} between **U** and **V** is given by

$$\rho_{UV} = \frac{\mathrm{E}\left[\left|V'_x(t)\right|^2 + \left|V'_y(t)\right|^2\right]}{\mathrm{E}\left[\left|V_x(t)\right|^2 + \left|V_y(t)\right|^2\right]}. \tag{18}$$

$\left(\left|V_x(t)\right|^2 + \left|V_y(t)\right|^2\right)^{1/2}$ is the length of $V(t)$ and $\left(\left|V'_x(t)\right|^2 + \left|V'_y(t)\right|^2\right)^{1/2}$ is the length of its estimation $V'(t)$. These quantities are independent of the chosen basis and, obviously, as expected, we have $\rho_{UV} \in [0,1]$, with

$$\left\{\begin{array}{c} \rho_{UV} = 1 \Longleftrightarrow V_x(t), V_y(t) \in \mathbf{H}_{U_x} + \mathbf{H}_{U_y} \\ \rho_{UV} = 0 \Longleftrightarrow V_x(t), V_y(t) \perp \mathbf{H}_{U_x} + \mathbf{H}_{U_y}. \end{array}\right.$$

As explained in section 2, the definition has no reason to be symmetric (generally $\rho_{UV} \neq \rho_{VU}$) and a more general index of coherence ρ_a can be defined:

$$\rho_a = a\rho_{UV} + (1-a)\,\rho_{VU} \tag{19}$$

where $a \in [0,1]$. Like in section 2, we remark that $\rho_{1/2}$ is a"symmetric index of coherence", and it is the only one provided that $\rho_{UV} \neq \rho_{VU}$.

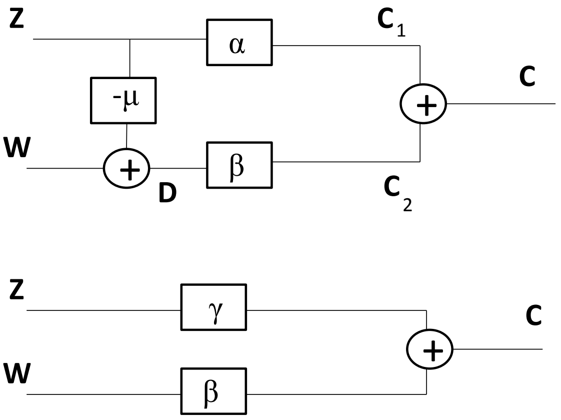

Fig. 4. LIF of complex gains $\alpha, \beta, \gamma, \mu$ defining $\mathbf{C}$ from $\mathbf{Z}$ and $\mathbf{W}$ when $C(t)$ belongs to $\mathbf{H}_Z + \mathbf{H}_W$.

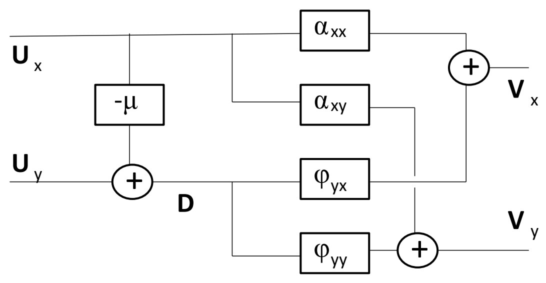

Fig. 5. circuit providing $\mathbf{V}_x$ and $\mathbf{V}_y$ from $\mathbf{U}_x$ and $\mathbf{V}_y$ when $V_x(t)$ and $V_y(t)$ belongs to $\mathbf{H}_{U_x} + \mathbf{H}_{V_y}$.

3.2. Estimation

As shown in the previous section, the best mean-square estimations of $V_x(t)$ and $V_y(t)$ from the observation of the processes $\mathbf{U}_x$ and $\mathbf{U}_y$ are $V_x'(t)$ and $V_y'(t)$ defined in (16) and (17) and which are the results of the device illustrated figure 5.

$$\left\{ \begin{array}{l} V_x'(t) = \mathcal{A}_{xx}[\mathbf{U}_x](t) + \mathcal{F}_{yx}[\mathbf{D}](t) \\ V_y'(t) = \mathcal{A}_{xy}[\mathbf{U}_x](t) + \mathcal{F}_{yy}[\mathbf{D}](t). \end{array} \right.$$

The estimation errors ε_x and ε_y are usually defined by

$$\left\{ \begin{array}{l} \varepsilon_x = \mathrm{E}\left[\left|V_x(t) - V_x'(t)\right|^2\right] \\ \varepsilon_y = \mathrm{E}\left[\left|V_y(t) - V_y'(t)\right|^2\right] \\ \frac{\varepsilon_x + \varepsilon_y}{\sigma_{V_x}^2 + \sigma_{V_y}^2} = 1 - \rho_{UV}. \end{array} \right.$$

Using (16) we obtain

$$\varepsilon_x = \int_{-\infty}^{\infty} \left[s_{V_x} - \frac{\left|s_{V_xU_x}\right|^2}{s_{U_x}} - \frac{\Delta\left|\phi_{yx}\right|^2}{s_{U_x}} \right](\omega)\,d\omega$$

$$\varepsilon_y = \int_{-\infty}^{\infty} \left[s_{V_y} - \frac{\left|s_{VyU_x}\right|^2}{s_{U_x}} - \frac{\Delta\left|\phi_{yy}\right|^2}{s_{U_x}} \right](\omega)\,d\omega \tag{20}$$

which leads to the formula

$$\begin{array}{c} \left(\sigma_{V_x}^2 + \sigma_{V_y}^2\right)\rho_{UV} = \\ \int_{-\infty}^{\infty} \left[\frac{|s_{V_xU_x}|^2}{s_{U_x}} + \frac{\Delta|\phi_{yx}|^2}{s_{U_x}}\right](\omega)\,d\omega + \\ \int_{-\infty}^{\infty} \left[\frac{|s_{VyU_x}|^2}{s_{U_x}} + \frac{\Delta|\phi_{yy}|^2}{s_{U_x}}\right](\omega)\,d\omega. \end{array} \tag{21}$$

Other formulas are available, replacing respectively in (16) $U_x, \phi_{yx}, \phi_{yy}$ by $U_y, \phi_{xy}, \phi_{xx}$ defined by

$$\left\{ \begin{array}{l} \phi_{xy} = \frac{1}{\Delta}\left[s_{V_yU_x}s_{U_y} - s_{V_yU_y}s_{U_yU_x}\right] \\ \phi_{xx} = \frac{1}{\Delta}\left[s_{V_xU_x}s_{U_y} - s_{V_xU_y}s_{U_yU_x}\right]. \end{array} \right.$$

3.3. Remark

In modern optics, the "spectral degree of coherence " $^\circ c(\omega)$ is defined by [2], [3]

$$^\circ c(\omega) = \left[\frac{s_{U_xV_x} + s_{U_yV_y}}{\sqrt{\left(s_{U_x} + s_{U_y}\right)\left(s_{V_x} + s_{V_y}\right)}} \right](\omega). \tag{22}$$

$^\circ c(\omega)$ is a complex quantity such that $0 \leq |^\circ c(\omega)| \leq 1$. The maximum value is obtained only when

$$\left\{ \begin{array}{c} s_{U_xV_x}(\omega) = \left[\sqrt{s_{U_x}s_{V_x}}\right](\omega) \\ s_{U_yV_y}(\omega) = \left[\sqrt{s_{U_y}s_{V_y}}\right](\omega) \\ \left[s_{U_x}/s_{U_y}\right](\omega) = \left[s_{V_x}/s_{V_y}\right](\omega) \end{array} \right. \tag{23}$$

which is a condition which separates the coordinates. Now, let assume that

$$U_x(t) = V_x(t), U_y(t) = 3V_y(t).$$

We verify that $^\circ c(\omega) < 1$. $\mathbf{U}$ and $\mathbf{V}$ are "coherent" in the sense where either of them defines the other (and by linear operations which can be infered). We find the same result when

$$U_x(t) = V_y(t), U_y(t) = 3V_x(t)$$

for $\omega \neq 2k\pi, k \in \mathbb{Z}$, though $\mathbf{U}$ and $\mathbf{V}$ are still "coherent". We have the same drawback for instance when

$$U_x(t) = V_y(t), U_y(t) = -V_x(t)$$

which can correspond to some rotation of a beam. In this case, we have $^\circ c(\omega) = 0$ when the processes $\mathbf{U}_x$ and $\mathbf{U}_y$ are uncorrelated, thought $\mathbf{U}$ defines $\mathbf{V}$ perfectly. We see through these simple examples that the notion of "coherence" that is used in this paper is different from the notion defined by (22).

3.4. Examples

3.4.1. Example 1

Let consider the simple model

$$\mathbf{V} = \mathbf{U} + \mathbf{N}$$

where $\mathbf{N}$=$(\mathbf{N}_x, \mathbf{N}_y)$ models an unpolarized beam, which means that

$$s_{N_x} = s_{N_y}, s_{N_x N_y} = 0$$

in any orthonormal basis [5], [6]. If $\mathbf{N}$ is uncorrelated with $\mathbf{U}$, we have, with respect to (15)

$$\mathbf{V}' = \mathbf{U}, \mathbf{V}'' = \mathbf{N}$$

Consequently (with $\sigma_{N_x} = \sigma_{N_y} = \sigma_N$)

$$\rho_{UV} = \frac{\sigma_{U_x}^2 + \sigma_{U_y}^2}{\sigma_{U_x}^2 + \sigma_{U_y}^2 + 2\sigma_N^2}.$$

ρ_{UV} decreases from 1 to 0 when σ_N increases from 0 to ∞, as expected.

When considering

$$\mathbf{U} = \mathbf{V} - \mathbf{N}$$

we no longer have

$$\mathbf{U}' = \mathbf{V}, \mathbf{U}'' = -\mathbf{N}$$

because, for instance

$$\mathrm{E}\left[N_x(t) V_x^*(u)\right] = \mathrm{E}\left[N_x(t) N_x^*(u)\right]$$

has no reason to cancel, and then we do not have

$$\mathbf{N}_x, \mathbf{N}_y \perp \mathbf{H}_{V_x} + \mathbf{H}_{V_y}.$$

The calculus of ρ_{VU} is tedious. We obtain, from (16)

$$\left\{\begin{array}{c} \sigma_{U_x'}^2 = \int_{-\infty}^{\infty} \left[\frac{s_N}{\Delta'}\left(\Delta + s_N s_{U_x}\right)\right](\omega)\, d\omega \\ \Delta' = s_{V_x} s_{V_y} - \left|s_{V_x V_y}\right|^2 \\ \Delta = s_{U_x} s_{U_y} - \left|s_{U_x U_y}\right|^2 \end{array}\right.$$

and $\sigma_{U_y'}^2$ by symmetry.

3.4.2. Example 2

We consider the model

$$\left\{\begin{array}{l} V_x(t) = U_x(t - X(t)) \\ V_y(t) = U_y(t - X(t)) \end{array}\right.$$

where $\mathbf{X}$ is defined section 2.4.2 by (13). This means that the propagation is delayed by a quantity which is random and identical for both components. The processes $\mathbf{V}_x$ and $\mathbf{V}_y$ can be decomposed following the sums [8]

$$\left\{\begin{array}{l} V_x(t) = G_x(t) + Y_x(t) \\ V_y(t) = G_y(t) + Y_y(t) \end{array}\right. \tag{24}$$

where $\mathbf{G}_x$ and $\mathbf{G}_y$ are the outputs of LIF with respective inputs $\mathbf{U}_x$ and $\mathbf{U}_y$ and the same complex gain $\alpha(\omega)$. $\mathbf{Y}_x$ is uncorrelated together with $\mathbf{G}_x$ and $\mathbf{G}_y$. The same property is true for $\mathbf{Y}_y$. Consequently,

$$\left\{\begin{array}{l} G_x(t), G_y(t) \in \mathbf{H}_{U_x} + \mathbf{H}_{U_y} \\ Y_x(t), Y_y(t) \perp \mathbf{H}_{U_x} + \mathbf{H}_{U_y}. \end{array}\right. \tag{25}$$

The power spectra s_{Y_x} and s_{Y_y} are different except when $s_{U_x} = s_{U_y}$. We find:

$$\rho_{UV} = \frac{1}{\sigma_{U_x}^2 + \sigma_{U_y}^2} \int_{-\infty}^{\infty} \left[|\alpha|^2 \left(s_{U_x} + s_{U_y}\right)\right](\omega)\, d\omega. \tag{26}$$

This result is consistent with intuition. For instance, if $\alpha(\omega)$ =sinc$[\theta\omega]$, characteristic function of a r.v. uniformly distributed on $(-\theta, \theta)$, the coherence is strong for small θ, i.e. for small variations of the propagation time, and the coherence will be weak for large deviations of the propagation time (and then for large θ). Computations are harder for ρ_{VU}, but are possible, knowing the Fourier transforms of

$$\left\{\begin{array}{c} K_{V_x}(\tau) = \int_{-\infty}^{\infty} e^{i\omega\tau} \beta(\tau, \omega)\, s_{U_x}(\omega)\, d\omega \\ K_{V_y}(\tau) = \int_{-\infty}^{\infty} e^{i\omega\tau} \beta(\tau, \omega)\, s_{U_y}(\omega)\, d\omega \\ K_{V_x V_y}(\tau) = \int_{-\infty}^{\infty} e^{i\omega\tau} \beta(\tau, \omega)\, s_{U_x U_y}(\omega)\, d\omega \\ K_{U_x V_x}(\tau) = \int_{-\infty}^{\infty} e^{i\omega\tau} \left[\alpha^* s_{U_x}\right](\omega)\, d\omega \\ K_{U_x V_y}(\tau) = \int_{-\infty}^{\infty} e^{i\omega\tau} \left[\alpha^* s_{U_x U_y}\right](\omega)\, d\omega. \end{array}\right.$$

3.4.3. Example 3

When $\mathbf{H}_{V_x}$+$\mathbf{H}_{V_y}$ is included in $\mathbf{H}_{U_x}$+$\mathbf{H}_{U_y}$, it is clear that $V_x(t)$ and $V_y(t)$ can be retrieved from $(\mathbf{U}_x, \mathbf{U}_y)$ which is equivalent to $\rho_{UV} = 1$. For instance, it is the case when

$$\left\{\begin{array}{l} V_x(t) = \mathcal{D}\left[U_x\right](t) + \mathcal{E}\left[U_y\right](t) \\ V_y(t) = \mathcal{F}\left[U_x\right](t) - \mathcal{G}\left[U_y\right](t) \end{array}\right. \tag{27}$$

where $\mathcal{D}$, $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}$ are well-defined LIF. Conversely, $\rho_{VU} = 1$ if and only when the linear system (27) can be inverted which is not always possible. For instance, let assume that the four filters are bandpass on $(-a, a)$. We have (with notations similar to section 3.1)

$$\left\{\begin{array}{l} U_x'(t) = \frac{1}{2}\left[V_x + V_y\right](t) \\ U_y'(t) = \frac{1}{2}\left[V_x - V_y\right](t) \end{array}\right.$$

which leads, by using (18), to

$$\rho_{VU} = \int_{-a}^{a} \frac{\left[s_{U_x} + s_{U_y}\right](\omega)}{\sigma_{U_x}^2 + \sigma_{U_y}^2} d\omega.$$

3.4.4. Example 4

We study the model

$$\left\{ \begin{array}{c} U_x(t) = X(t) + M_x(t), U_y(t) = X(t) + M_y(t) \\ V_x(t) = Y(t) + N_x(t), V_y(t) = Y(t) + N_y(t) \end{array} \right.$$

where **M**=$\{\mathbf{M}_x, \mathbf{M}_y\}$,**N**=$\{\mathbf{N}_x, \mathbf{N}_y\}$ are unpolarized (see section 3.4.1) and uncorrelated between them and with $(\mathbf{X}, \mathbf{Y})$. The power spectral densities are s_X, s_Y, s_M and s_N. We find, using (16) and (20)

$$\begin{array}{c} \rho_{UV} = \frac{2}{\sigma_Y^2+\sigma_N^2} \int_{-\infty}^{\infty} \left[\frac{|s_{YX}|^2}{2s_X+s_M}\right](\omega)\, d\omega \\ \rho_{VU} = \frac{2}{\sigma_X^2+\sigma_M^2} \int_{-\infty}^{\infty} \left[\frac{|s_{YX}|^2}{2s_Y+s_N}\right](\omega)\, d\omega \end{array}$$

The result verifies (8), when $\sigma_N = \sigma_M = 0$. We obtain the same ρ_{UV} with

$$\left\{ \begin{array}{c} U_x(t) = X(t)(\cos\theta - \sin\theta) + M_x(t) \\ U_y(t) = X(t)(\cos\theta + \sin\theta) + M_y(t) \\ V_x(t) = Y(t)(\cos\theta - \sin\theta) + N_x(t), \\ V_y(t) = Y(t)(\cos\theta + \sin\theta) + N_y(t) \end{array} \right.$$

for any θ, following the properties of invariance by rotation.

3.4.5. Example 5

Finally, let $\mathbf{V}_x, \mathbf{V}_y$ be two real processes, and $\mathbf{U}_x, \mathbf{U}_y$ the corresponding analytic signals [10]. This means that for instance (the integral is defined in the Cauchy sense)

$$U_x(t) = V_x(t) + i \int_{-\infty}^{\infty} \frac{V_x(u)}{\pi(t-u)} du.$$

We know that the analytic signal loses the negative part of the power spectrum and we easily find the formulas

$$\left\{ \begin{array}{c} s_{U_x}(\omega) = 4s_{V_x}(\omega), s_{U_y}(\omega) = 4s_{V_y}(\omega) \\ s_{U_xU_y}(\omega) = 4s_{V_xV_y}(\omega), \omega > 0 \end{array} \right.$$

and 0 for $\omega < 0$. Obviously, $\rho_{VU} = 1$. But $V_x(t)$ and $V_y(t)$ are the real parts of $U_x(t)$ and $U_y(t)$ and then the former (real) processes can be deduced from the latter (complex) processes. Nevertheless, the operation which transforms a complex function in its real part is not linear, which explains why $\rho_{UV} < 1$. Actually, we find $\rho_{UV} = 1/2$, using (16) and (18). This last example shows the limitations of linear tools, as explained section 2.4.4.

4. CONCLUSION

The coherence of a field can be defined as a measure of the proximity between some properties measured at two points of the field. If the field is reduced to only one random variable at each point of the space, a correlation coefficient depending on coordinates of any couple of points may be a good measure of coherence, the values 0 and 1 addressing the lack of dependence and, conversely, the complete dependence. When the field is characterized by one-dimensional stationary processes (for instance **X** and **Y** at two points), the normalized cross-correlation (1) is a natural measure of dependence of $X(t-\tau)$ on $Y(t)$, though the latter may be influenced by the entire set of the $X(u), u \in \mathbb{R}$, and not only by the value $X(t-\tau)$ of **X** at $t-\tau$. If the entire process **X** is observed, and assuming the stationarity property, formula (1) does not provide the whole available information held by **X** about the elements of **Y**. A positive number which measures global links between $X(t)$ and the entire process **Y** (or the converse) appears to be a better characterization of proximity of both processes. It seems equivalent looking for a copy $\widetilde{\mathbf{X}}$ of **X** from **Y** (an estimation of $X(t)$ on the entire set of the $Y(u)$) and conversely, which is an usual procedure in communications. Obviously, the idea of coherence is linked to the similarity between the model and the copy. At the next stage, we compute a distance (the error) between both and we define an index of coherence normalizing the latter. We have explained the main drawback of this construction: it leads to a different index when **X** and **Y** are inverted. Nevertheless, this enables the definition of an available linear family of indices. We show that other constructions can be achieved which lead to a symmetric index of coherence. When the field is no longer one-dimensional but two-dimensional, definitions are generalized. The main idea is unchanged, which looks for characterizing a kind of distance between Hilbert spaces respectively spanned by each of two-dimensional processes. The resulting "index of coherence" is still a number between 0 and 1 as expected and not some function of time or frequency. Examples are given to cover a sufficient number of situations, and appendices summarize the main results of the stationary process theory, and detail laborious calculations.

5. APPENDICES

5.1. Appendix 1: notations

1) Let **U**=$\{U(t), t \in \mathbb{R}\}$ be a zero-mean stationary process. Auto-correlation function K_U, spectral density s_U and total power σ_U^2 verify

$$\begin{array}{c} K_U(\tau) = \mathrm{E}\left[U(t)\, U^*(t-\tau)\right] = \int_{-\infty}^{\infty} s_U(\omega)\, e^{i\omega\tau} d\omega \\ \sigma_U^2 = K_U(0). \end{array}$$

where E[..] and the superscript * stand for the mathematical expectation (ensemble mean) the complex conjugate.

2) The cross-correlation K_{UV}, the cross-spectral density s_{UV} between the processes **U** and **V** are defined by

$$K_{UV}(\tau) = \mathrm{E}\left[U(t)V^*(t-\tau)\right] = \int_{-\infty}^{\infty} s_{UV}(\omega) e^{i\omega\tau} d\omega$$

when both processes are stationary and have stationary cross-correlations (equivalently $(\mathbf{U}, \mathbf{V})$ is stationary). All these quantities are always assumed regular enough.

3) $\mathbf{H}_U$ is the Hilbert space of linear combinations of the $U(t), t \in \mathbb{R}$. This means that (for some $t_{kn} \in \mathbb{R}$, $a_{kn} \in \mathbb{C}$)

$$A \in \mathbf{H}_U \iff A = \lim_{n\to\infty} \sum_{k=-n}^{n} a_{kn} U(t_{kn})$$

in the mean-square sense. The scalar product $\langle ., . \rangle_{\mathbf{H}_U}$ in $\mathbf{H}_U$ is defined by

$$\langle A, B \rangle_{\mathbf{H}_U} = \mathrm{E}\left[AB^*\right].$$

4) $\mathbf{K}_{s_U}$ is the Hilbert space of complex valued functions f such that

$$\int_{-\infty}^{\infty} \left[|f|^2 s_U\right](\omega) d\omega < \infty.$$

The scalar product $\langle ., . \rangle_{\mathbf{K}_{s_U}}$ is defined by

$$\langle f, g \rangle_{\mathbf{K}_{s_U}} = \int_{-\infty}^{\infty} \left[f g^* s_U\right](\omega) d\omega.$$

5) The isometry $\mathbf{I}_U$ between $\mathbf{H}_U$ and $\mathbf{K}_{s_U}$ is defined from the correspondence

$$U(t) \Longleftrightarrow_{\mathbf{I}_U} e^{i\omega t}.$$

If $A = \lim_{n\to\infty} \sum_{k=-n}^{n} a_{kn} U(t_{kn})$, then

$$A \Longleftrightarrow_{\mathbf{I}_U} \lim_{n\to\infty} \sum_{k=-n}^{n} a_{kn} e^{i\omega t_{kn}}.$$

Moreover, if $A \Longleftrightarrow_{\mathbf{I}_U} \alpha, B \Longleftrightarrow_{\mathbf{I}_U} \beta$, then

$$\mathrm{E}\left[|A-B|^2\right] = \int_{-\infty}^{\infty} \left[|\alpha - \beta|^2 s_U\right](\omega) d\omega.$$

The isometry allows to solve a problem of distance between random variables (r.v.) using Fourier analysis.

6) The Linear Invariant Filter (LIF) $\mathcal{F}$ with complex gain ϕ, input $U(t)$, output $V(t)$ is defined by

$$V(t) = \mathcal{F}[\mathbf{U}](t) \Longleftrightarrow_{\mathbf{I}_U} \phi(\omega) e^{i\omega t}.$$

The impulse response f of $\mathcal{F}$ is defined by (in some sense)

$$\phi(\omega) = \int_{-\infty}^{\infty} f(u) e^{-i\omega u} du.$$

For a regular enough f, we have

$$\mathcal{F}[\mathbf{U}](t) = \int_{-\infty}^{\infty} f(u) U(t-u) du.$$

If $W(t) = \mathcal{G}[\mathbf{U}](t)$ is the output of the LIF of complex gain γ, we have

$$\mathrm{E}\left[V(t) W^*(t-\tau)\right] = \int_{-\infty}^{\infty} \left[\phi\gamma^* s_U\right](\omega) e^{i\omega\tau} d\omega$$
$$s_{VW}(\omega) = \left[\phi\gamma^* s_U\right](\omega).$$

This relation is known as the "theorem of interferences".

Though the principles above are very general, we assume that the used processes have bounded spectral densities. Nevertheless, results in this paper are true for monochromatic waves, which are approximations of waves encountered in the real word.

5.2. Appendix 2

Let assume that $A(t)$ is the orthogonal projection of $V(t)$ on $\mathbf{H}_U$:

$$A(t) = \mathrm{pr}_{\mathbf{H}_U} V(t).$$

This means that $V(t) - A(t)$ is orthogonal to any $U(u)$ (the r.v. which generate $\mathbf{H}_U$) :

$$\mathrm{E}\left[(V(t) - A(t)) U^*(u)\right] = 0$$

for any $u \in \mathbb{R}$. Equivalently, whatever u

$$\int_{-\infty}^{\infty} \left(s_{VU}(\omega) e^{i\omega(t-u)} - \left[\phi_t s_U\right](\omega) e^{-i\omega u}\right) d\omega = 0$$

when, in the usual isometry $\mathbf{I}_U$ built from $\mathbf{H}_U$, we have the correspondences

$$U(t) \longleftrightarrow_{\mathbf{I}_U} e^{i\omega t}, A(t) \longleftrightarrow_{\mathbf{I}_U} \phi_t(\omega).$$

As a consequence of the unicity of the Fourier transform, we deduce

$$\phi_t(\omega) = \left[\frac{s_{VU}}{s_U}\right](\omega) e^{i\omega t}$$

which means that **A** is the output of the LIF with input **U** and complex gain

$$\phi(\omega) = \left[\frac{s_{VU}}{s_U}\right](\omega).$$

In the equality (6), $B(t)$ is orthogonal to $\mathbf{H}_U$ and then orthogonal together to the $U(u)$ and the $A(u)$.

5.3. Appendix 3

We consider three stationary processes **Z, W, C.** We assume that $(\mathbf{Z}, \mathbf{W})$ is stationary, that **C** has stationary correlations with $(\mathbf{Z}, \mathbf{W})$ and that $C(t) \in \mathbf{H}_Z + \mathbf{H}_W$ (which means that $C(t)$ is the result of linear operations from **Z** and **W**). We have to justify the drawings of figure 4, where $\mu(\omega), \alpha(\omega), \beta(\omega), \gamma(\omega)$ are complex gains of LIF to be characterized. s_Z, s_{ZW}... are the spectral densities and the cross-spectra.

1) Let $\mathbf{C}_1$ be defined by

$$C_1(t) = \mathrm{pr}_{\mathbf{H}_Z} C(t).$$

If $C_1(t) \longleftrightarrow_{I_Z} \alpha_t(\omega)$, we have (whatever $t, u \in \mathbb{R}$)

$$\begin{gathered} \mathrm{E}\left[\left(C(t) - C_1(t)\right) Z^*(u)\right] = 0 \Longleftrightarrow \\ \int_{-\infty}^{\infty} e^{-i\omega u}\left[e^{i\omega t} s_{CZ}(\omega) - \left[\alpha_t s_Z\right](\omega)\right] d\omega = 0 \end{gathered}$$

which is equivalent to

$$\alpha_t(\omega) = e^{i\omega t}\left[\frac{s_{CZ}}{s_Z}\right](\omega).$$

Consequently, $\mathbf{C}_1$ is the output of a LIF with complex gain

$$\alpha(\omega) = \left[\frac{s_{CZ}}{s_Z}\right](\omega). \tag{28}$$

2) Let $\mathbf{D}$ be defined by

$$D(t) = W(t) - \mathcal{M}[\mathbf{Z}](t)$$

where $\mathcal{M}$ is some LIF complex gain $\mu(\omega)$. We look for μ such that $D(t) \perp \mathbf{H}_Z$:

$$\begin{gathered} \mathrm{E}\left[\left(W(t) - \mu[\mathbf{Z}](t)\right) Z^*(u)\right] = 0 \Longleftrightarrow \\ \int_{-\infty}^{\infty} e^{i\omega(t-u)}\left[s_{WZ} - \mu s_Z\right](\omega)\, d\omega = 0 \end{gathered}$$

which yields

$$\mu(\omega) = \left[\frac{s_{WZ}}{s_Z}\right](\omega). \tag{29}$$

Moreover, we remark that

$$\mathbf{H}_Z + \mathbf{H}_W = \mathbf{H}_Z + \mathbf{H}_D$$

because $W(t) = D(t) + \mathcal{M}[\mathbf{Z}](t)$. Also

$$\begin{gathered} s_D(\omega) = \left[s_W - \frac{|s_{ZW}|^2}{s_Z}\right](\omega) \\ s_{DC}(\omega) = \left[s_{WC} - \frac{s_{WZ} s_{ZC}}{s_Z}\right](\omega). \end{gathered} \tag{30}$$

3) Let $\mathbf{C}_2$ defined by

$$C_2(t) = \mathrm{pr}_{\mathbf{H}_D} C(t).$$

By construction, $\mathbf{H}_Z$ and $\mathbf{H}_D$ are orthogonal and $\mathbf{H}_C \subset \mathbf{H}_Z$ $+\mathbf{H}_W$ by hypothesis, which implies

$$C(t) = C_1(t) + C_2(t).$$

The problem is to prove that $\mathbf{C}_2$ is the output of a LIF. If $C_2(t) \longleftrightarrow_{I_D} \beta_t(\omega)$, we have

$$\begin{gathered} \mathrm{E}\left[\left(C(t) - C_2(t)\right) D^*(u)\right] = 0 \Longleftrightarrow \\ \int_{-\infty}^{\infty} e^{-i\omega u}\left[e^{i\omega t}\phi(\omega) - \left[\beta_t s_D\right](\omega)\right] d\omega = 0. \\ \phi(\omega) = \left[s_{CW} - \mu^* s_{CZ}\right](\omega) \end{gathered}$$

Using (29) and (30) we obtain

$$\beta_t(\omega) = e^{i\omega t}\left[\frac{s_{CW} s_Z - s_{CZ} s_{ZW}}{s_W s_Z - |s_{ZW}|^2}\right](\omega)$$

which proves that $\mathbf{C}_2$ is the output of a LIF $\mathcal{B}$ with input $\mathbf{D}$ and complex gain

$$\beta(\omega) = \left[\frac{s_{CW} s_Z - s_{CZ} s_{ZW}}{s_W s_Z - |s_{ZW}|^2}\right](\omega). \tag{31}$$

Figure 4 depicts a symmetric equivalent circuit which highlights the LIF of complex gain $\gamma(\omega)$ with

$$\gamma(\omega) = \left[\frac{s_{CZ} s_W - s_{CW} s_{WZ}}{s_W s_Z - |s_{ZW}|^2}\right](\omega). \tag{32}$$

As a consequence, the power spectrum s_C verifies

$$\begin{gathered} s_C(\omega) = \\ \left[\frac{|s_{CZ}|^2 s_W + |s_{CW}|^2 s_Z - 2\mathcal{R}\left[s_{CW} s_{WZ} s_{ZC}\right]}{s_W s_Z - |s_{WZ}|^2}\right](\omega). \end{gathered}$$

Moreover, the symmetric scheme is unique, provided that the set of ω such that

$$|s_{ZW}|(\omega) \neq \sqrt{s_Z s_W}(\omega)$$

has a positive measure.

4) If $\mathbf{C}$ and $\mathbf{C}'$ have stationary correlations with $(\mathbf{Z}, \mathbf{W})$ and belong to $\mathbf{H}_Z + \mathbf{H}_W$, then $(\mathbf{C}, \mathbf{C}')$ is stationary and with cross-spectrum

$$\left\{\begin{gathered} s_{CC'}(\omega) = \\ \left[\frac{a s_{CZ} - b s_{CW}}{s_W s_Z - |s_{WZ}|^2}\right](\omega) \\ a(\omega) = \left[s_W s_{ZC'} - s_{ZW} s_{WC'}\right](\omega) \\ b(\omega) = \left[s_Z s_{WC'} - s_{WZ} s_{ZC'}\right](\omega). \end{gathered}\right.$$

To summarize, if $\mathbf{C}$ and $\mathbf{C}'$ are stationarily correlated with $\mathbf{Z}$ and $\mathbf{W}$ and belong to $\mathbf{H}_Z + \mathbf{H}_W$, they are the outputs of a "bi-filter" with perfectly determined components.

5) When the hypothesis $C(t) \in \mathbf{H}_Z + \mathbf{H}_W$ is suppressed, we have the decomposition

$$\begin{gathered} C(t) = C'(t) + C''(t) \\ C'(t) = \mathrm{pr}_{\mathbf{H}_Z + \mathbf{H}_W} C(t) \\ C''(t) \perp \mathbf{H}_Z + \mathbf{H}_W. \end{gathered}$$

$\mathbf{C}'$ is stationarily correlated with $(\mathbf{Z}, \mathbf{W})$ because

$$\mathrm{E}\left[C'(t)\left[aZ + bW\right]^*(u)\right] = \left[aK_{CZ} + bK_{CW}\right](t-u).$$

As a consequence, the drawings in figure 4 are available, replacing $\mathbf{C}$ by $\mathbf{C}'$ as output (but with the same complex gains $\mu, \alpha, \beta, \gamma$).

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